Pairing correlations

- Two-particle propagator with 2 times
- Defined in the medium and in free space

- In free space —> scattering and bound states
- Develop from diagrams relevant equations

- What happens in the homogeneous medium
- Cooper problem
- Pairing instability

Two-particle propagator

- Here we consider states in N±2
- Collective effects associated with these states pertain to pairing
- But proper treatment also incorporates short-range correlations associated with repulsive cores…
- Relevant diagrams: so-called ladder diagrams
- Transform bare interaction in free space to T-matrix
- In-medium: additional effects Pauli and dispersion

Two-time two-particle propagator

- As usual, two times means only one energy variable
- Definition $G_{pphh}(\alpha,\alpha';\beta,\beta';t_1-t_2) \equiv \lim_{t_1'\to t_1}\ \lim_{t_2'\to t_2} G_{II}(\alpha t_1,\alpha' t_1',\beta t_2,\beta' t_2')$

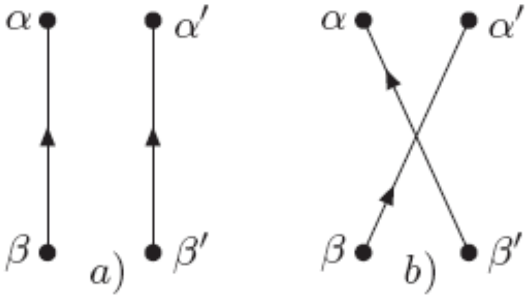
$$= -\frac{i}{\hbar}\langle \Psi_0^N|\,\mathcal{T}[a_{\alpha'_H}(t_1)a_{\alpha_H}(t_1)a^\dagger_{\beta_H}(t_2)a^\dagger_{\beta'_H}(t_2)]\,|\Psi_0^N\rangle$$

- Label pphh emphasizes possibility of adding or removing pairs
- Noninteracting propagator directly from Wick's theorem

$$G^{(0)}_{pphh}(\alpha,\alpha';\beta,\beta';t_1-t_2) = -\frac{i}{\hbar}\langle \Phi_0^N|\,\mathcal{T}[a_{\alpha'}(t_1)a_{\alpha}(t_1)a^\dagger_{\beta}(t_2)a^\dagger_{\beta'}(t_2)]\,|\Phi_0^N\rangle$$

$$= i\hbar\left[G^{(0)}(\alpha,\beta;t_1-t_2)G^{(0)}(\alpha',\beta';t_1-t_2) - G^{(0)}(\alpha,\beta';t_1-t_2)G^{(0)}(\alpha',\beta;t_1-t_2)\right]$$

- Diagrams



- Consider only mean-field single-particle propagators for now

Noninteracting tp propagator

- Allows use of diagonal sp propagators (HF in finite system)

$$G^{(0)}(\alpha,\alpha';t_1-t_2)\equiv\delta_{\alpha,\alpha'}G^{(0)}(\alpha;t_1-t_2)$$

- So we can write

$$G^{(0)}_{pphh}(\alpha,\alpha';\beta,\beta';t_1-t_2)=i\hbar\,[\delta_{\alpha,\beta}\delta_{\alpha',\beta'}-\delta_{\alpha,\beta'}\delta_{\alpha',\beta}]G^{(0)}(\alpha;t_1-t_2)G^{(0)}(\alpha';t_1-t_2)$$

- Energy formulation

$$\begin{aligned}G^{(0)}_{pphh}(\alpha,\alpha';\beta,\beta';E)&=\int_{-\infty}^{\infty}d(t_1-t_2)\;e^{\frac{i}{\hbar}E(t_1-t_2)}G^{(0)}_{pphh}(\alpha,\alpha';\beta,\beta';t_1-t_2)\\&=i\hbar\,[\delta_{\alpha,\beta}\delta_{\alpha',\beta'}-\delta_{\alpha,\beta'}\delta_{\alpha',\beta}]\int_{-\infty}^{\infty}d(t_1-t_2)\;e^{\frac{i}{\hbar}E(t_1-t_2)}\\&\times\int_{-\infty}^{\infty}\frac{dE_1}{2\pi\hbar}e^{-iE_1(t_1-t_2)/\hbar}\;G^{(0)}(\alpha;E_1)\int_{-\infty}^{\infty}\frac{dE_2}{2\pi\hbar}e^{-iE_2(t_1-t_2)/\hbar}\;G^{(0)}(\alpha';E_2)\\&=i\hbar\,[\delta_{\alpha,\beta}\delta_{\alpha',\beta'}-\delta_{\alpha,\beta'}\delta_{\alpha',\beta}]\int_{-\infty}^{\infty}\frac{dE_1}{2\pi\hbar}G^{(0)}(\alpha;E_1)G^{(0)}(\alpha';E-E_1)\end{aligned}$$

- Evaluate integral

$$\begin{aligned}G^{(0)}_{pphh}(\alpha,\alpha';\beta,\beta';E)&=[\delta_{\alpha,\beta}\delta_{\alpha',\beta'}-\delta_{\alpha,\beta'}\delta_{\alpha',\beta}]\left\{\frac{\theta(\alpha-F)\theta(\alpha'-F)}{E-\varepsilon_\alpha-\varepsilon_{\alpha'}+i\eta}-\frac{\theta(F-\alpha)\theta(F-\alpha')}{E-\varepsilon_\alpha-\varepsilon_{\alpha'}-i\eta}\right\}\\&\equiv[\delta_{\alpha,\beta}\delta_{\alpha',\beta'}-\delta_{\alpha,\beta'}\delta_{\alpha',\beta}]\;G^{(0)}_{pphh}(\alpha,\alpha';E)\end{aligned}$$

First-order term

- Consider first-order term without self-energy terms

$$G^{(1)}_{pphh}(\alpha,\alpha';\beta,\beta';t_1-t_2) = \left(\frac{-i}{\hbar}\right)^2 \int dt\, \frac{1}{4} \sum_{\gamma\gamma'\delta\delta'} \langle \gamma\gamma'|\,V\,|\delta\delta'\rangle$$

$$\langle \Phi^N_0|\,\mathcal{T}\left[a^\dagger_\gamma(t)a^\dagger_{\gamma'}(t)a_{\delta'}(t)a_\delta(t)a_{\alpha'}(t_1)a_\alpha(t_1)a^\dagger_\beta(t_2)a^\dagger_{\beta'}(t_2)\right]|\Phi^N_0\rangle$$

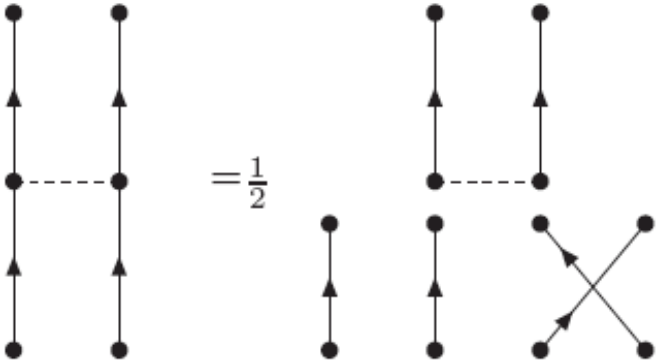
$$\Rightarrow (i\hbar)^2 \int dt \sum_{\gamma\gamma'\delta\delta'} \langle\gamma\gamma'|\,V\,|\delta\delta'\rangle\, G^{(0)}(\alpha,\gamma;t_1-t)G^{(0)}(\alpha',\gamma';t_1-t)G^{(0)}(\delta,\beta;t-t_2)G^{(0)}(\delta',\beta';t-t_2)$$

- FT in various forms

$$G^{(1)}_{pphh}(\alpha,\alpha';\beta,\beta';E) = G^{(0)}_{pphh}(\alpha,\alpha';E)\ \langle\alpha\alpha'|\,V\,|\beta\beta'\rangle\ G^{(0)}_{pphh}(\beta,\beta';E)$$

$$= G^{(0)}_{pphh}(\alpha,\alpha';E)\ \frac{1}{2}\sum_{\gamma\gamma'}\langle\alpha\alpha'|\,V\,|\gamma\gamma'\rangle\ G^{(0)}_{pphh}(\gamma,\gamma';\beta,\beta';E)$$

- Graphically



Ladders in free space

- Iteration of the interaction to all orders obtained by replacing last noninteracting propagator by interacting one
- First consider two particles in free space (no holes) and

$$\begin{aligned} |\Psi_0^N\rangle &\rightarrow |0\rangle \\ |\Phi_0^N\rangle &\rightarrow |0\rangle \end{aligned}$$

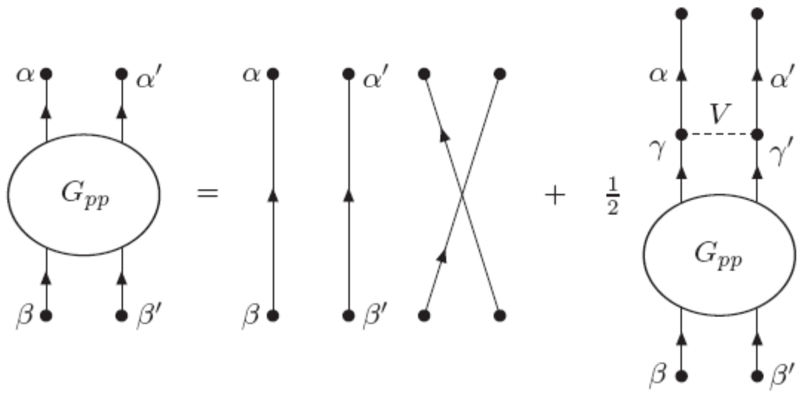
- Notation (note: no step functions)

$$G_{pp}^{(0)}(\alpha,\alpha';\beta,\beta';E) = [\delta_{\alpha,\beta}\delta_{\alpha',\beta'} - \delta_{\alpha,\beta'}\delta_{\alpha',\beta}] \left\{ \frac{1}{E - \varepsilon_\alpha - \varepsilon_{\alpha'} + i\eta} \right\}$$

- Ladder summation (same in the medium but with $G_{pphh}^{(0)}$)

$$\begin{aligned} G_{pp}(\alpha,\alpha';\beta,\beta';E) &= G_{pp}^{(0)}(\alpha,\alpha';\beta,\beta';E) \\ &+ G_{pp}^{(0)}(\alpha,\alpha';E)\ \frac{1}{2}\sum_{\gamma\gamma'} \langle\alpha\alpha'|\, V\, |\gamma\gamma'\rangle\ G_{pp}(\gamma,\gamma';\beta,\beta';E) \end{aligned}$$

- Diagrams



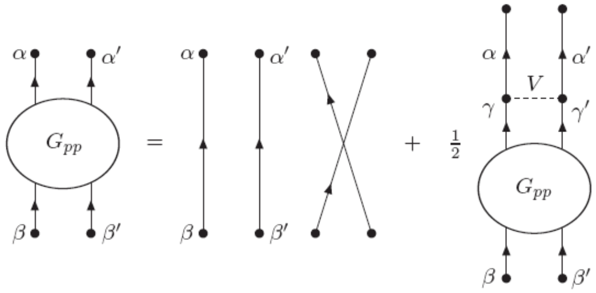
there are no other ones!

Free space

- No other diagrams generated in free space (need holes)
- Medium: ladder diagrams treat short-range correlations but there are other diagrams (including self-energy corrections)
- Factor $\frac{1}{2}$:
  - each (antisymmetrized) V yields $\frac{1}{4}$
  - each noninteracting propagator either 2 or 4 quantum numbers
  - for 2 quantum numbers: symmetry of interaction yields factor of 2 (so first-order yields $\frac{1}{4}$ X 4 =1)

$$G^{(1)}_{pphh}(\alpha,\alpha';\beta,\beta';E) = G^{(0)}_{pphh}(\alpha,\alpha';E)\ \langle\alpha\alpha'|\,V\,|\beta\beta'\rangle\ G^{(0)}_{pphh}(\beta,\beta';E)$$
$$= G^{(0)}_{pphh}(\alpha,\alpha';E)\ \frac{1}{2}\sum_{\gamma\gamma'}\langle\alpha\alpha'|\,V\,|\gamma\gamma'\rangle\ G^{(0)}_{pphh}(\gamma,\gamma';\beta,\beta';E)$$

  - nth order: $(\frac{1}{4})^n \times 2^{n+1}$
  - factor of $\frac{1}{2}$ in integral equation automatically takes care of this

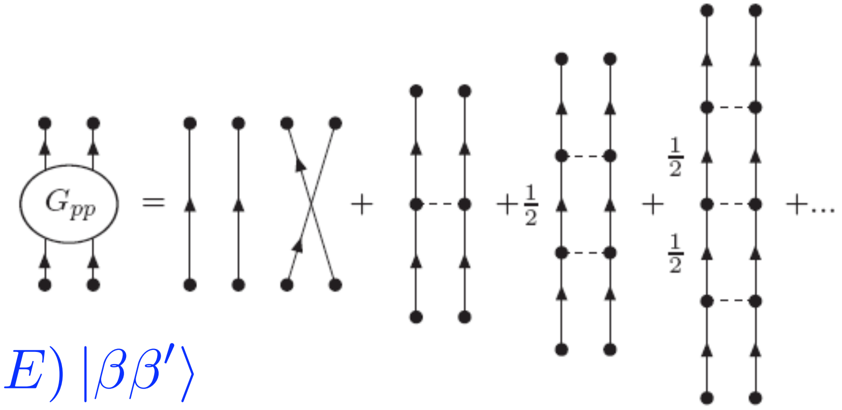


Alternative summation

- Arrange summation according to

$$G_{pp}(\alpha,\alpha';\beta,\beta';E) = G^{(0)}_{pp}(\alpha,\alpha';\beta,\beta';E) + G^{(0)}_{pp}(\alpha,\alpha';E)\ \langle\alpha\alpha'|\,\Gamma_{pp}(E)\,|\beta\beta'\rangle\ G^{(0)}_{pp}(\beta,\beta';E)$$

- accordingly



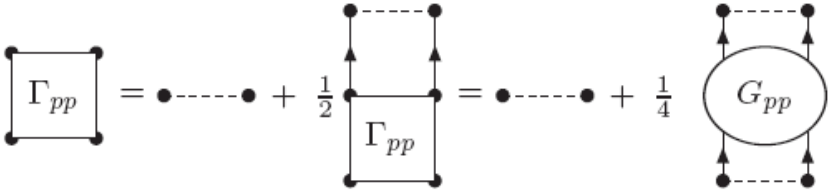
- where $\langle\alpha\alpha'|\,\Gamma_{pp}(E)\,|\beta\beta'\rangle$

$$= \langle\alpha\alpha'|\,V\,|\beta\beta'\rangle + \frac{1}{2}\sum_{\gamma\gamma'}\langle\alpha\alpha'|\,V\,|\gamma\gamma'\rangle\ G^{(0)}_{pp}(\gamma,\gamma';E)\ \langle\gamma\gamma'|\,\Gamma_{pp}(E)\,|\beta\beta'\rangle$$

- Also

$$\langle\alpha\alpha'|\,\Gamma_{pp}(E)\,|\beta\beta'\rangle$$

$$= \langle\alpha\alpha'|\,V\,|\beta\beta'\rangle + \frac{1}{4}\sum_{\gamma\gamma'}\sum_{\delta\delta'}\langle\alpha\alpha'|\,V\,|\gamma\gamma'\rangle\ G_{pp}(\gamma,\gamma';\delta,\delta';E)\ \langle\delta\delta'|\,V\,|\beta\beta'\rangle$$

- Diagrams  -->Poles of G_{pp} and Γ_{pp} same

Scattering of two particles in free space

- Ladder summation usually referred to as $\mathcal{T}$-matrix
- Use wave vectors (momentum)
- Conserved total wave vector $\boldsymbol{K} = \boldsymbol{k}_\alpha + \boldsymbol{k}_{\alpha'} = \boldsymbol{k}_\beta + \boldsymbol{k}_{\beta'}$
- Relative wave vectors (final, initial, intermediate)

$$\begin{aligned} \boldsymbol{k} &= \tfrac{1}{2}\left(\boldsymbol{k}_\alpha - \boldsymbol{k}_{\alpha'}\right) \\ \boldsymbol{k}' &= \tfrac{1}{2}\left(\boldsymbol{k}_\beta - \boldsymbol{k}_{\beta'}\right) \\ \boldsymbol{q} &= \tfrac{1}{2}\left(\boldsymbol{k}_\gamma - \boldsymbol{k}_{\gamma'}\right) \end{aligned}$$

- Transcribe

$$\begin{aligned} &\langle \alpha\alpha' |\, \Gamma_{pp}(E)\, | \beta\beta' \rangle \\ &\quad = \langle \alpha\alpha' |\, V\, | \beta\beta' \rangle + \frac{1}{2} \sum_{\gamma\gamma'} \langle \alpha\alpha' |\, V\, | \gamma\gamma' \rangle\ G^{(0)}_{pp}(\gamma,\gamma';E)\ \langle \gamma\gamma' |\, \Gamma_{pp}(E)\, | \beta\beta' \rangle \end{aligned}$$

- to

$$\begin{aligned} &\langle \boldsymbol{k} m_\alpha m_{\alpha'} |\, \Gamma_{pp}(\boldsymbol{K},E)\, | \boldsymbol{k}' m_\beta m_{\beta'} \rangle = \langle \boldsymbol{k} m_\alpha m_{\alpha'} |\, V\, | \boldsymbol{k}' m_\beta m_{\beta'} \rangle \\ &\quad + \frac{1}{2} \sum_{m_\gamma m_{\gamma'}} \int \frac{d^3q}{(2\pi)^3}\ \langle \boldsymbol{k} m_\alpha m_{\alpha'} |\, V\, | \boldsymbol{q} m_\gamma m_{\gamma'} \rangle\, G^{(0)}_{pp}(\boldsymbol{K},\boldsymbol{q};E)\, \langle \boldsymbol{q} m_\gamma m_{\gamma'} |\, \Gamma_{pp}(\boldsymbol{K},E)\, | \boldsymbol{k}' m_\beta m_{\beta'} \rangle \end{aligned}$$

- volume terms cancel

Simple considerations

- Noninteracting propagator (spin/isospin considered for matrix elements of interaction)

$$G^{(0)}_{pp}(\boldsymbol{K},\boldsymbol{q};E) = \frac{1}{E - \varepsilon(\frac{1}{2}\boldsymbol{K}+\boldsymbol{q}) - \varepsilon(\frac{1}{2}\boldsymbol{K}-\boldsymbol{q}) + i\eta}$$

- with

$$\varepsilon(\boldsymbol{k}) = \frac{\hbar^2\boldsymbol{k}^2}{2m}$$

- Isolate available energy in the center of mass

$$E = \frac{\hbar^2\boldsymbol{K}^2}{4m} + E_0 \equiv \frac{\hbar^2\boldsymbol{K}^2}{4m} + \frac{\hbar^2 k_0^2}{m}$$

- Since

$$\varepsilon(\tfrac{1}{2}\boldsymbol{K}+\boldsymbol{q}) + \varepsilon(\tfrac{1}{2}\boldsymbol{K}-\boldsymbol{q}) = \frac{\hbar^2\boldsymbol{K}^2}{4m} + \frac{\hbar^2\boldsymbol{q}^2}{m}$$

- there is no dependence on the center-of-mass wave vector (drop)
- Not the case in the medium

Partial-wave decomposition

- For short-range interactions a partial-wave decomposition is practical
- For nucleon-nucleon (NN) scattering

$$\langle k\ell|\,\Gamma_{pp}^{JST}(k_0)\,|k'\ell'\rangle = \langle k\ell|\,V^{JST}\,|k'\ell'\rangle + \frac{m}{2\hbar^2}\sum_{\ell''}\int\frac{dq\ q^2}{(2\pi)^3}\ \langle k\ell|\,V^{JST}\,|q\ell''\rangle\,\frac{1}{k_0^2-q^2+i\eta}\,\langle q\ell''|\,\Gamma_{pp}^{JST}(k_0)\,|k'\ell'\rangle$$

- Relation between on-shell matrix element and phase shift for uncoupled channel

$$\langle k_0\ell|S^{JST}(k_0)|k_0\ell\rangle = \left[1-2\pi i\left(\frac{mk_0}{2\hbar^2}\right)\langle k_0\ell|\Gamma_{pp}^{JST}(k_0)|k_0\ell\rangle\right] \equiv e^{2i\delta_\ell^{JST}}$$

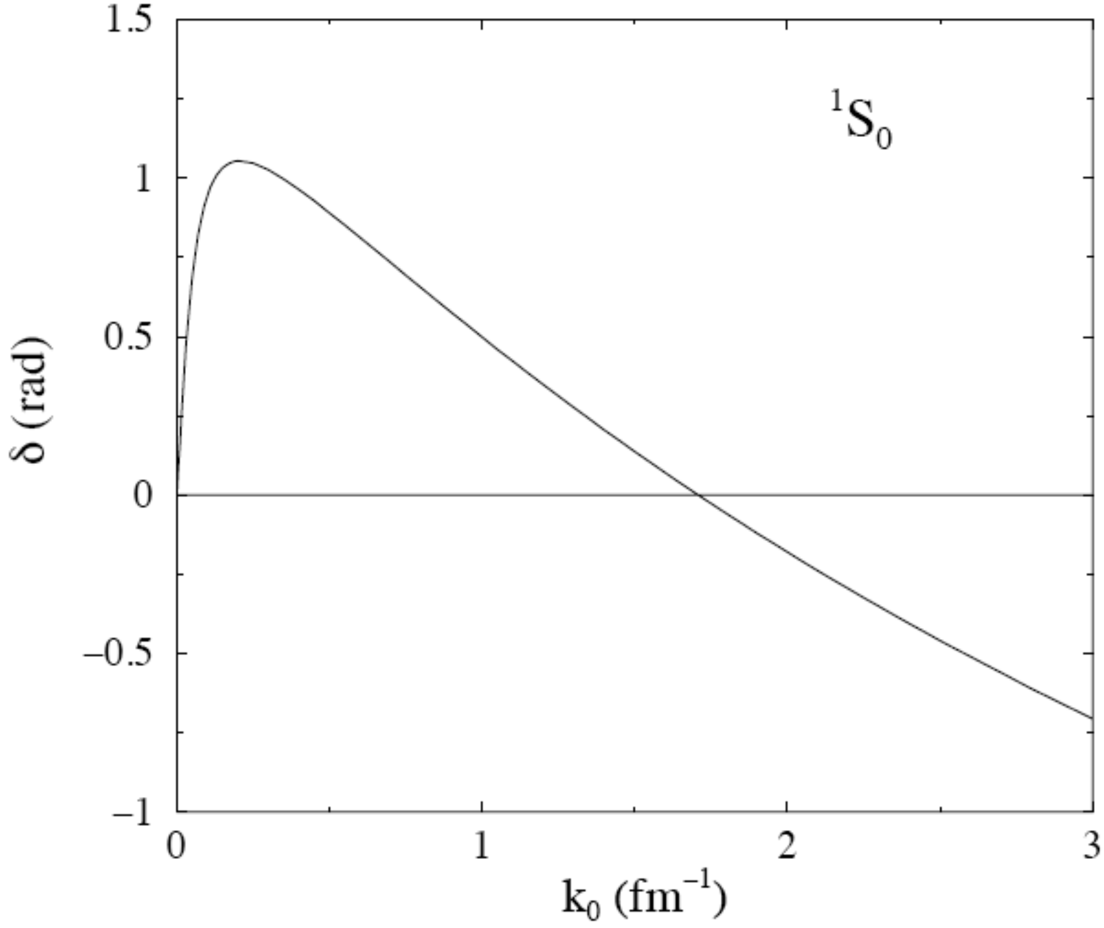
- Equivalent to

$$\tan\delta_\ell^{JST} = \frac{\mathrm{Im}\ \langle k_0\ell|\Gamma_{pp}^{JST}(k_0)|k_0\ell\rangle}{\mathrm{Re}\ \langle k_0\ell|\Gamma_{pp}^{JST}(k_0)|k_0\ell\rangle}$$

- so nonvanishing imaginary part required for nonzero phase shift

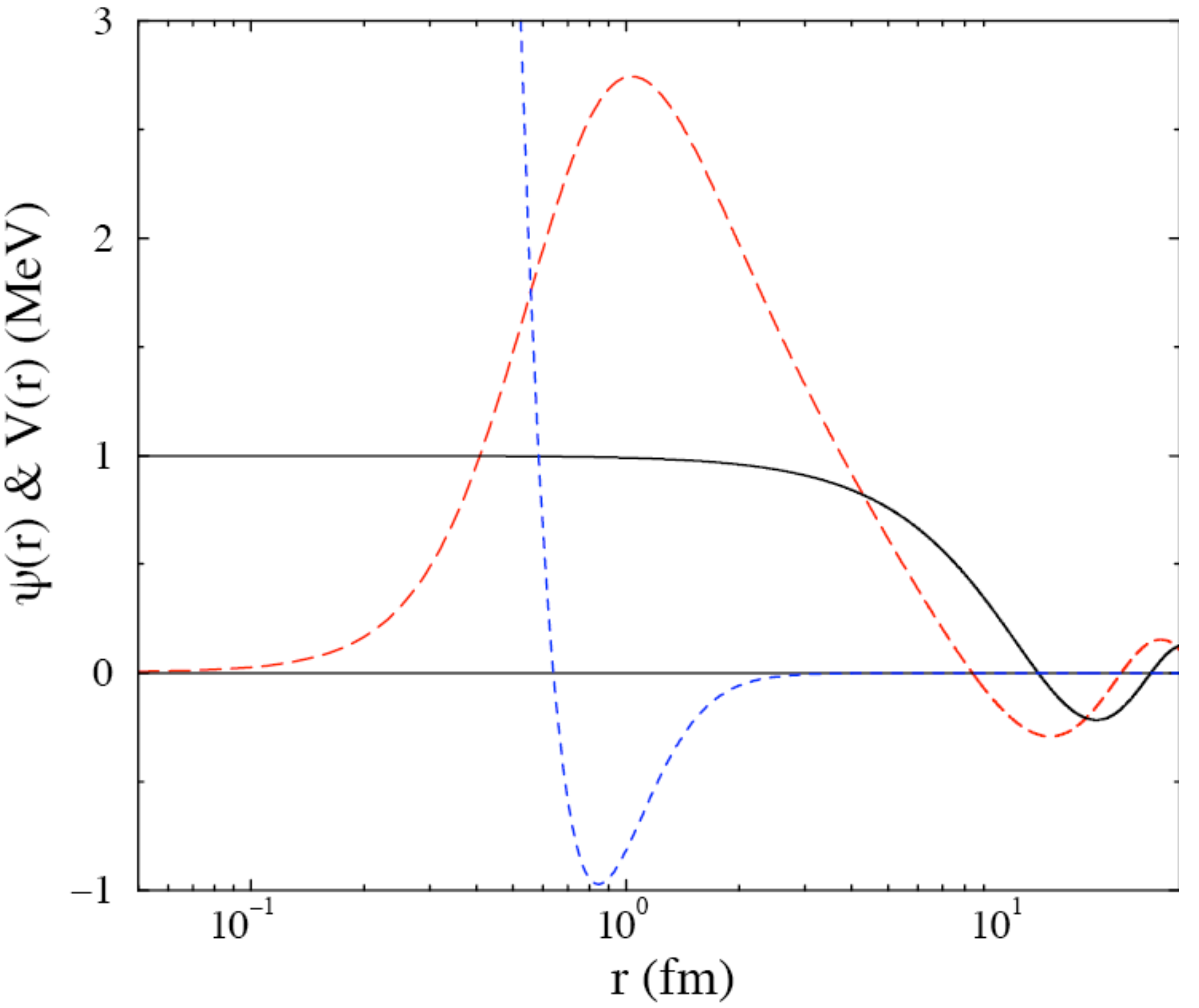
Some results

- Discretize integration --> matrix inversion
- Already sufficient only to iterate principal value part (R-matrix)
- $^1S_0$ phase shift from Reid-soft-core NN interaction
- Attraction at low energy
- Repulsion at higher energy



Visualize effect of summation

- Scattering energy --> $k_0 = 0.25\ \mathrm{fm}^{-1}$
- Free wave function - solid
- Correlated - long dashes
- Potential/100 - short dashes
- Note logarithmic scale horizontal axis
- Correlated wave function disappears where interaction strongly repulsive



Coupled channels

- Asymptotic analysis not much more involved (2x2 for NN)
- Includes nondiagonal orbital angular momentum term on account of tensor terms (but total spin must be 1)
- Corresponding S-matrix

$$\langle k_0\ell|\mathcal{S}^{JST}(k_0)|k_0\ell'\rangle = \left[\delta_{\ell,\ell'} - 2\pi i\left(\frac{mk_0}{2\hbar^2}\right)\langle k_0\ell|\Gamma^{JST}_{pp}(k_0)|k_0\ell'\rangle\right]$$

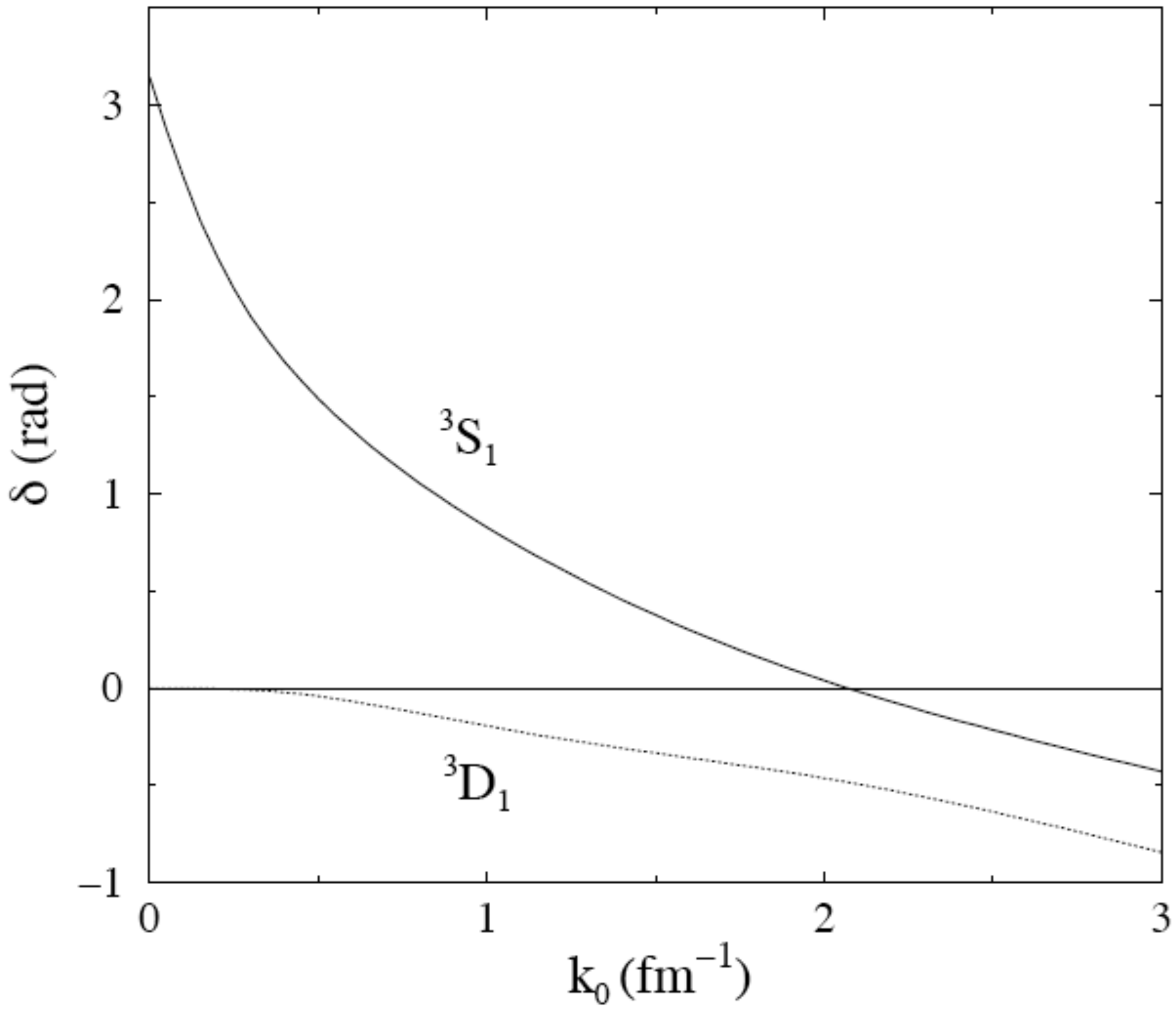
- S unitary allows diagonalization by orthogonal real matrix according to

$$\langle k_0\ell|\mathcal{S}^{JST}(k_0)|k_0\ell'\rangle = \sum_{\alpha=1,2}\langle\ell|A^J(k_0)|\alpha\rangle e^{2i\delta^{JST}_\alpha}\langle\alpha|A^J(k_0)|\ell'\rangle$$

- with eigenphase shifts δ^{JST}_α
- Convention $\langle\ell|A^J(k_0)|\alpha\rangle = \begin{pmatrix}\cos\epsilon^J & \sin\epsilon^J \\ -\sin\epsilon^J & \cos\epsilon^J\end{pmatrix}$
- with mixing angle and mixing parameter $\rho^J = \sin 2\epsilon^J$
- Three real parameters characterize elastic scattering

Example

- $^3S_1$-$^3D_1$ coupled channel has a bound state (deuteron)
- One phase shift must start at π
- Considerable mixing
- Still old notation



Bound states of two particles

- Lehmann representation of tp propagator

$$
\begin{aligned}
G_{pphh}(\alpha,\alpha';\beta,\beta';E) = \sum_m \frac{\langle\Psi_0^N|\, a_{\alpha'} a_\alpha \,|\Psi_m^{N+2}\rangle \,\langle\Psi_m^{N+2}|\, a_\beta^\dagger a_{\beta'}^\dagger \,|\Psi_0^N\rangle}{E-(E_m^{N+2}-E_0^N)+i\eta} \\
+\int_{\varepsilon_T^+}^{\infty} d\tilde{E}_\mu^{N+2} \frac{\langle\Psi_0^N|\, a_{\alpha'} a_\alpha \,|\Psi_\mu^{N+2}\rangle \,\langle\Psi_\mu^{N+2}|\, a_\beta^\dagger a_{\beta'}^\dagger \,|\Psi_0^N\rangle}{E-\tilde{E}_\mu^{N+2}+i\eta} \\
-\sum_n \frac{\langle\Psi_0^n|\, a_\beta^\dagger a_{\beta'}^\dagger \,|\Psi_n^{N-2}\rangle \,\langle\Psi_n^{N-2}|\, a_{\alpha'} a_\alpha \,|\Psi_0^N\rangle}{E-(E_0^N-E_n^{N-2})-i\eta} \\
-\int_{-\infty}^{\varepsilon_T^-} d\tilde{E}_\nu^{N-2} \frac{\langle\Psi_0^N|\, a_\beta^\dagger a_{\beta'}^\dagger \,|\Psi_\nu^{N-2}\rangle \,\langle\Psi_\nu^{N-2}|\, a_{\alpha'} a_\alpha \,|\Psi_0^N\rangle}{E-\tilde{E}_\nu^{N-2}-i\eta}
\end{aligned}
$$

- Note possible discrete states and continuum thresholds
- Covers the medium case
- No N-2 states for free particles
- --> Reference state vacuum

Development

- Bound state for two particles in free space $|\Psi_n^{N=2}\rangle = |\boldsymbol{K} n\rangle$ includes cm wave vector
- n labels intrinsic quantum numbers
- For $\boldsymbol{K} = 0$ we identify numerator of Lehmann rep

$$\langle 0|\, a_{-\boldsymbol{k}m_{\alpha'}} a_{\boldsymbol{k}m_\alpha} \,|\boldsymbol{K} = 0\ n\rangle = \langle \boldsymbol{K} = 0\ \boldsymbol{k}; m_\alpha m_{\alpha'} |\boldsymbol{K} = 0\ n\rangle = \psi_n(\boldsymbol{k}; m_\alpha m_{\alpha'})$$

- as wave function (in relative wave vector) of bound state
- Eigenvalue problem from propagator equation: standard
- Poles for bound states in interacting propagator; only branch cut for positive energy for noninteracting propagator

$$\frac{\hbar^2 \boldsymbol{k}^2}{m} \psi_n(\boldsymbol{k}; m_\alpha m_{\alpha'}) + \frac{1}{2} \sum_{m_\gamma m_{\gamma'}} \int \frac{d^3 q}{(2\pi)^3} \langle \boldsymbol{k} m_\alpha m_{\alpha'}|\, V \,|\boldsymbol{q} m_\gamma m_{\gamma'}\rangle\, \psi_n(\boldsymbol{q}; m_\gamma m_{\gamma'})$$

$$= E_n\ \psi_n(\boldsymbol{k}; m_\alpha m_{\alpha'})$$

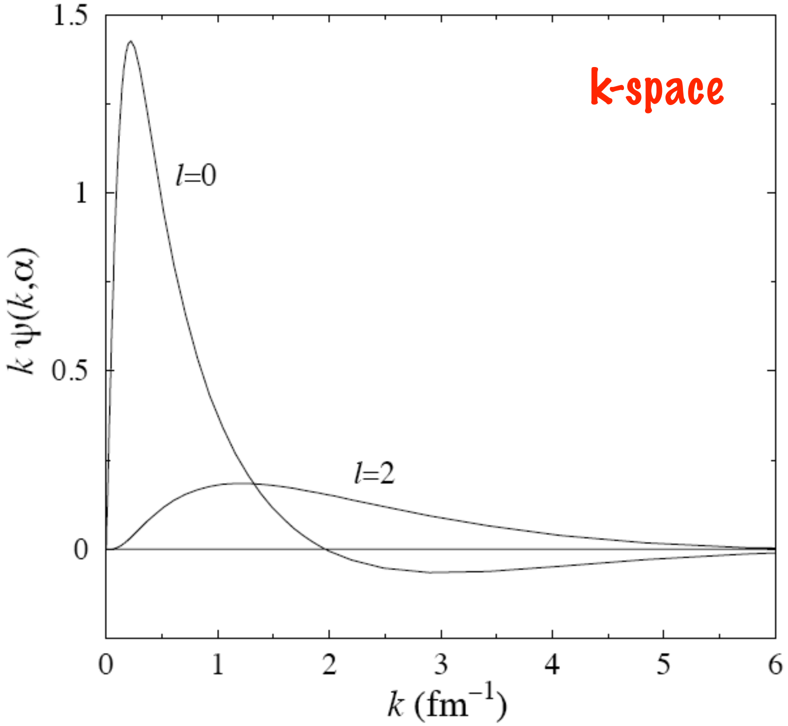
Deuteron

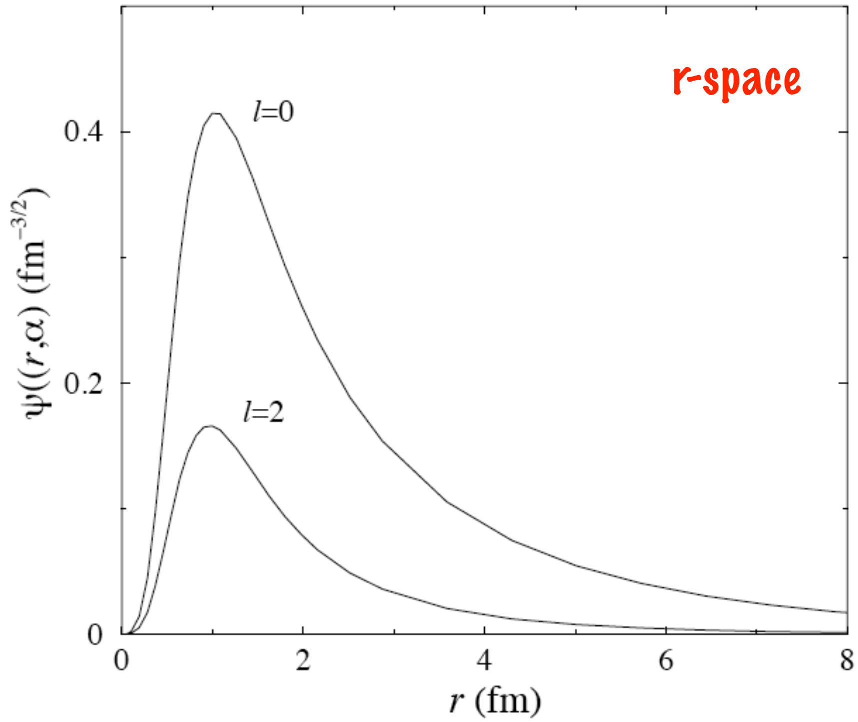
- Rotational invariance, parity, etc. and partial wave decomposition combined with coupling to total spin and isospin

$$\frac{\hbar^2 k^2}{m}\psi_n(k(\ell S)JT) + \frac{1}{2}\sum_{\ell'}\int\frac{dq\ q^2}{(2\pi)^3}\left\langle k\ell\right| V^{JST}\left|q\ell'\right\rangle \psi_n(q(\ell' S)JT)$$

$$= E_n\ \psi_n(k(\ell S)JT)$$

- Deuteron Reid potential 6.5% D-state





$$\psi_n(r(\ell S)JT) = \sqrt{\frac{2}{\pi}}\int_0^\infty dk k^2\ \mathrm{j}_\ell(kr)\ \psi_n(k(\ell S)JT)$$

Ladder diagrams and SRC in the medium

- Ladder diagrams take care of SRC
- Preserved in the medium
- Concentrate on solution of ladder equation in the medium with mean-field sp propagators but including hh term: (more later)

$$\langle \boldsymbol{k} m_\alpha m_{\alpha'} | \Gamma(\boldsymbol{K}, E) | \boldsymbol{k}' m_\beta m_{\beta'} \rangle = \langle \boldsymbol{k} m_\alpha m_{\alpha'} | V | \boldsymbol{k}' m_\beta m_{\beta'} \rangle$$
$$+ \frac{1}{2} \sum_{m_\gamma m_{\gamma'}} \int \frac{d^3q}{(2\pi)^3} \; \langle \boldsymbol{k} m_\alpha m_{\alpha'} | V | \boldsymbol{q} m_\gamma m_{\gamma'} \rangle \, G^{(0)}_{pphh}(\boldsymbol{K}, \boldsymbol{q}; E) \, \langle \boldsymbol{q} m_\gamma m_{\gamma'} | \Gamma(\boldsymbol{K}, E) | \boldsymbol{k}' m_\beta m_{\beta'} \rangle$$

- with

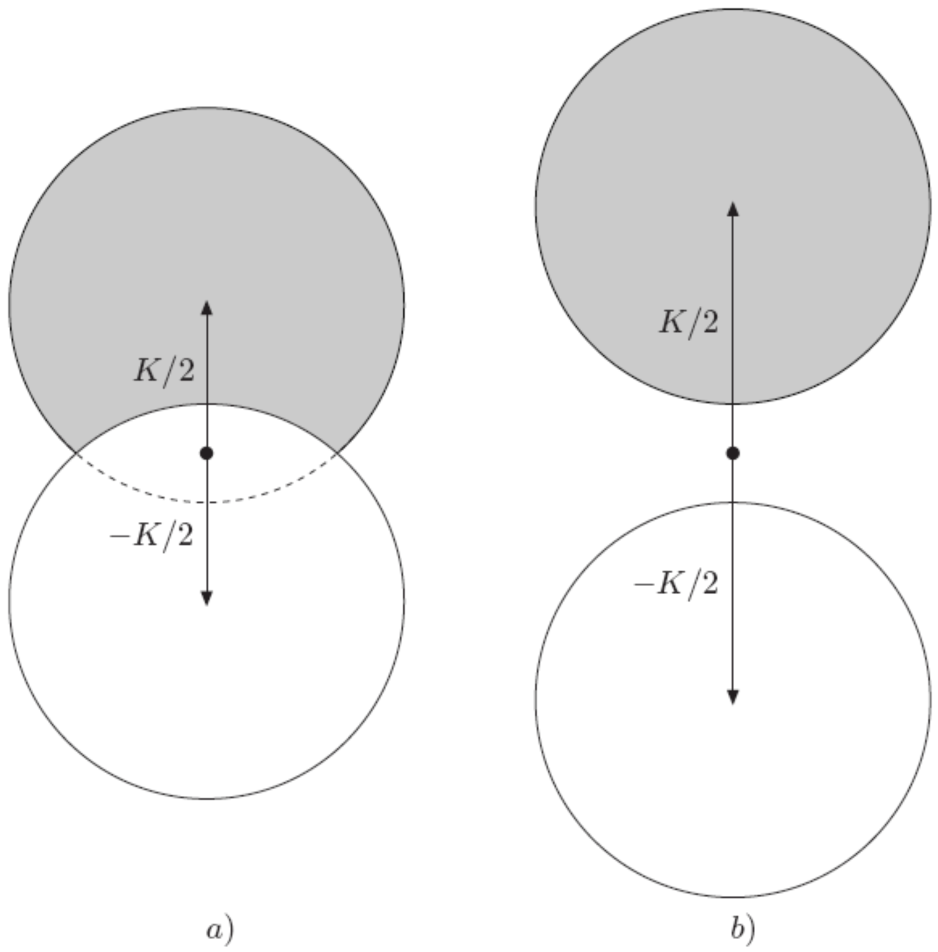
$$G^{(0)}_{pphh}(\boldsymbol{K}, \boldsymbol{q}; E) = \frac{\theta(|\boldsymbol{K}/2 + \boldsymbol{q}| - k_F)\, \theta(|\boldsymbol{K}/2 - \boldsymbol{q}| - k_F)}{E - \varepsilon(\boldsymbol{K}/2 + \boldsymbol{q}) - \varepsilon(\boldsymbol{K}/2 - \boldsymbol{q}) + i\eta} - \frac{\theta(k_F - |\boldsymbol{K}/2 + \boldsymbol{q}|)\, \theta(k_F - |\boldsymbol{K}/2 - \boldsymbol{q}|)}{E - \varepsilon(\boldsymbol{K}/2 + \boldsymbol{q}) - \varepsilon(\boldsymbol{K}/2 - \boldsymbol{q}) - i\eta}$$

- can also be written as

$$G^{(0)}_{pphh}(\boldsymbol{K}, \boldsymbol{q}; E) = i \int \frac{dE'}{2\pi} \; G^{(0)}(\boldsymbol{K}/2 + \boldsymbol{q}; E/2 + E') G^{(0)}(\boldsymbol{K}/2 - \boldsymbol{q}; E/2 - E')$$

Phase space and Pauli principle

- Introduces total wave vector dependence illustrated in figure
- a) total wave vector < $2k_F$
- b) $>2k_F$
- Constraint by step functions
- Outside both spheres: pp
- Inside both: hh
- Most phase space for $|\mathbf{K}|=0$
- Extremely relevant for possible bound states...

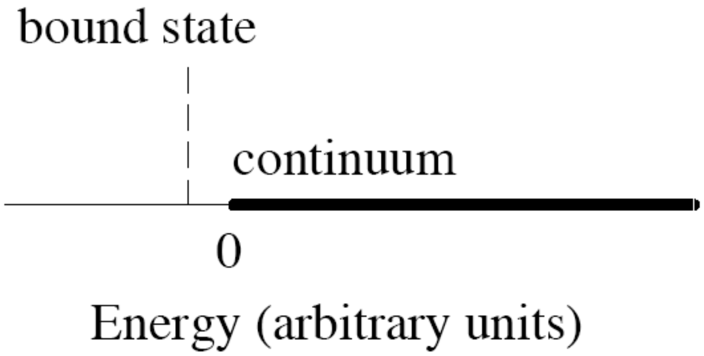


Appearance of bound-pair states & Cooper problem

- Reminder of appearance of bound states for free particles
- Rewrite eigenvalue equation in wave vector space

$$\psi_n(\boldsymbol{k};m_\alpha m_{\alpha'}) = \frac{1}{E_n - \hbar^2\boldsymbol{k}^2/m}\,\frac{1}{2}\sum_{m_\gamma m_{\gamma'}}\int\frac{d^3q}{(2\pi)^3}\,\langle \boldsymbol{k} m_\alpha m_{\alpha'}|\,V\,|\boldsymbol{q} m_\gamma m_{\gamma'}\rangle\,\psi_n(\boldsymbol{q};m_\gamma m_{\gamma'})$$

- Two electrons or two ^{3}He atoms with spin $\frac{1}{2}$ have antisymmetry requirement $\ell + S$ even
- For $\ell = 0$ spin $S = 0$
- For $\ell = 1$ spin $S = 1$ and so on
- In this basis $\psi_n(k;\ell S) = \dfrac{1}{E_n - \hbar^2\boldsymbol{k}^2/m}\,\dfrac{1}{2}\displaystyle\int\frac{dq\ q^2}{(2\pi)^3}\,\langle k|\,V^{\ell S}\,|q\rangle\,\psi_n(q;\ell S)$
- Visualize appearance of bound state

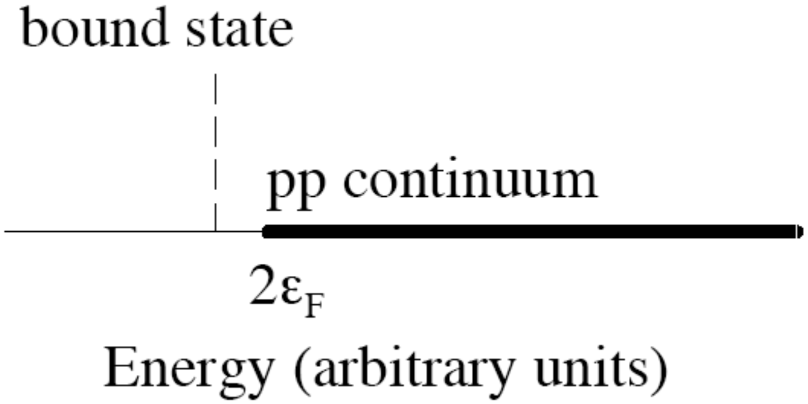


In the medium

- Two particles on top of the Fermi sea
- Most favorable total wave vector --> zero

$$G_{pp}^{(0)}(\boldsymbol{K}=0,q;E)=\frac{\theta(q-k_F)}{E-2\varepsilon(q)+i\eta}$$

- Similar to free space



- Eigenvalue equation

$$\psi_C(k;\ell S)=\frac{\theta(k-k_F)}{E_C-2\varepsilon(k)}\ \frac{1}{2}\int\frac{dq\ q^2}{(2\pi)^3}\left\langle k\right|V^{\ell S}\left|q\right\rangle\psi_C(q;\ell S)$$

- Subscript C for Cooper
- Use separable interaction to illustrate properties

Cooper problem

- Interaction $\langle k|\, V^{\ell S}\, |q\rangle = \lambda_\ell w_\ell(k) w_\ell^*(q)$
- S implied
- Substitute —> $\psi_C(k;\ell S) = \mathcal{N} \dfrac{\theta(k-k_F)w_\ell(k)}{E_C - 2\varepsilon(k)}$
- with $\mathcal{N} = \frac{1}{2}\lambda_\ell \int \frac{dq\ q^2}{(2\pi)^3} w_\ell^*(q)\psi_C(q;\ell S)$
- Amplitude substituted in eigenvalue equation yields

$$\frac{1}{\lambda_\ell} = \frac{1}{2}\int \frac{dq\ q^2}{(2\pi)^3}\frac{\theta(q-k_F)|w_\ell(q)|^2}{E_C - 2\varepsilon(q)}$$

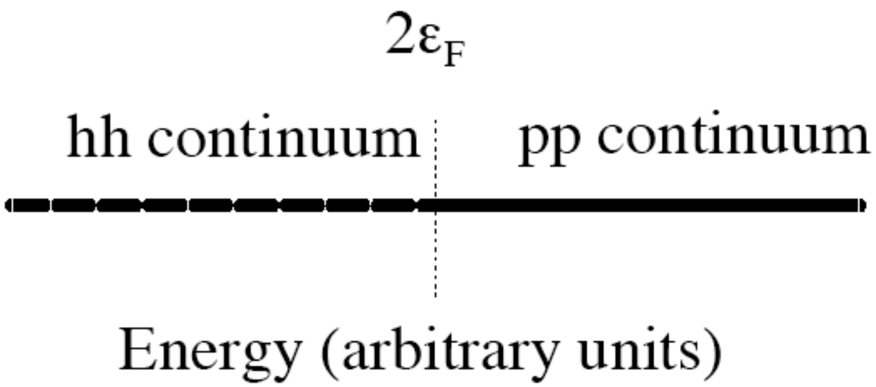
- Right side negative definite for energy below pp continuum, diverging to -∞ when approaching this limit
- So always solution for attractive interaction!
- None for repulsive interaction
- Peculiarity: bound state resides in hh continuum...

Inclusion of hh propagation

- Attempt to include hh propagation in eigenvalue equation

$$\begin{aligned}\psi_C(k;\ell S) &= \frac{\theta(k-k_F)}{E_C-2\varepsilon(k)}\,\frac{1}{2}\int\frac{dq\;q^2}{(2\pi)^3}\left\langle k\right|V^{\ell S}\left|q\right\rangle\psi_C(q;\ell S)\\ &- \frac{\theta(k_F-k)}{E_C-2\varepsilon(k)}\,\frac{1}{2}\int\frac{dq\;q^2}{(2\pi)^3}\left\langle k\right|V^{\ell S}\left|q\right\rangle\psi_C(q;\ell S)\end{aligned}$$

- Visualize unperturbed spectrum
- No "room" for bound states
- Either pp or hh



- Not possible to have discrete (real) eigenvalues for an attractive interaction
- Instead yields complex eigenvalues signaling instability of starting point (pairing instability)

Bound-pair states

- Consider original propagator equation $G^{(0)}_{pphh}(\boldsymbol{K}=0,q;E) = \frac{\theta(q-k_F)}{E-2\varepsilon(q)+i\eta} - \frac{\theta(k_F-q)}{E-2\varepsilon(q)-i\eta}$

$$G^{\ell S}_{pphh}(k,k';E) = G^{(0)}_{pphh}(k,k';E)$$
$$+\ G^{(0)}_{pphh}(k;E)\ \frac{1}{2}\int \frac{dq\ q^2}{(2\pi)^3}\ \langle k|\, V^{\ell S}\, |q\rangle\ \ G^{\ell S}_{pphh}(q;k';E)$$

- Cannot legitimately eliminate noninteracting propagator
- Unless there is a GAP in the sp spectrum at k_F
- Add auxiliary potential with a constant shift Δ below k_F
- Implies gap of 2Δ between pp and hh continuum
- Now a legitimate eigenvalue problem can be obtained
- Use separable interaction to get transition amplitudes

$$\psi_{BP}(k;\ell S) = \mathcal{N}\frac{\theta(k-k_F)w_\ell(k)}{E_{BP}-2\varepsilon(k)} \qquad \psi_{BP}(k;\ell S) = -\mathcal{N}\frac{\theta(k_F-k)w_\ell(k)}{E_{BP}-2\varepsilon(k)}$$

- and eigenvalue problem

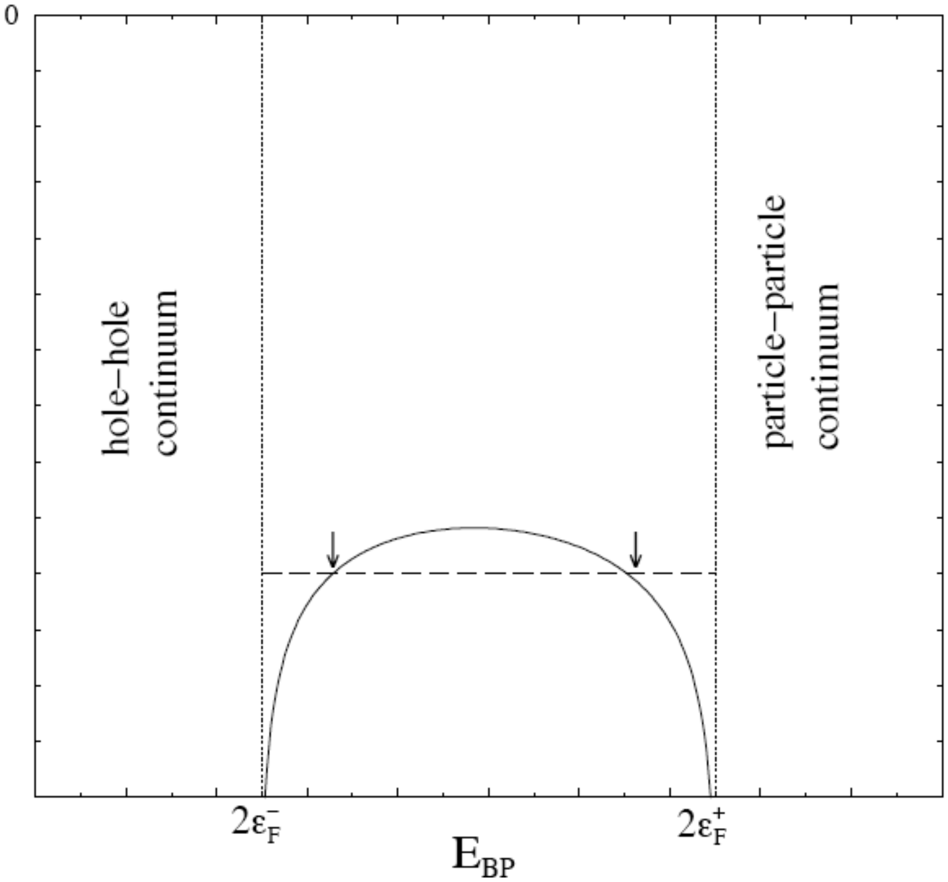
$$\frac{1}{\lambda_\ell} = \frac{1}{2}\int\frac{dq\ q^2}{(2\pi)^3}\,\frac{\theta(q-k_F)|w_\ell(q)|^2}{E_{BP}-2\varepsilon(q)} - \frac{1}{2}\int\frac{dq\ q^2}{(2\pi)^3}\,\frac{\theta(k_F-q)|w_\ell(q)|^2}{E_{BP}-2\varepsilon(q)}$$

Graphical illustration

- Plot right side of

$$\frac{1}{\lambda_\ell} = \frac{1}{2}\int \frac{dq\ q^2}{(2\pi)^3}\frac{\theta(q-k_F)|w_\ell(q)|^2}{E_{BP}-2\varepsilon(q)} - \frac{1}{2}\int \frac{dq\ q^2}{(2\pi)^3}\frac{\theta(k_F-q)|w_\ell(q)|^2}{E_{BP}-2\varepsilon(q)}$$

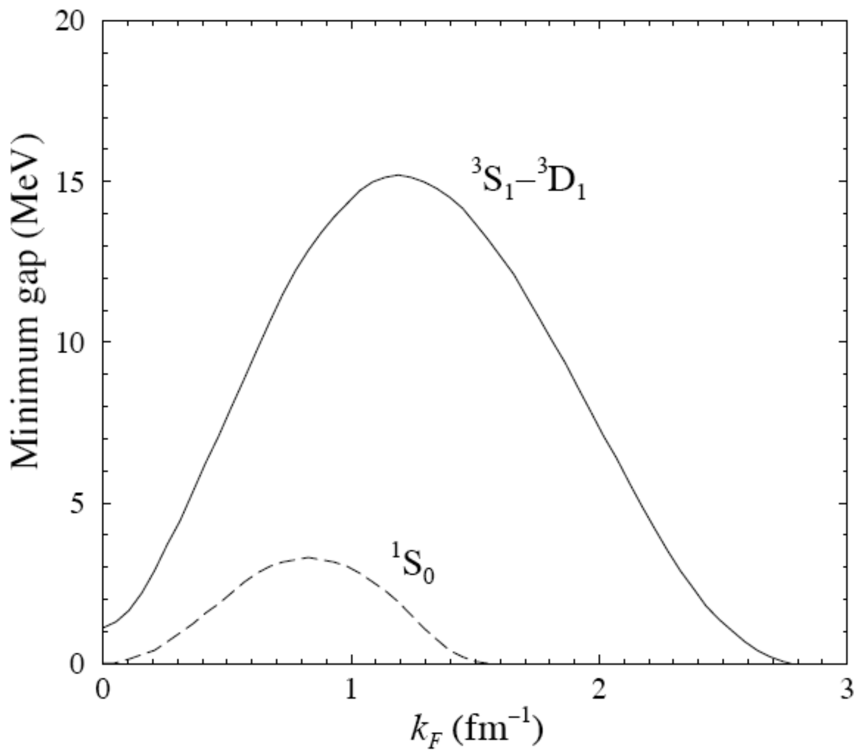
- as a function of E_{BP} between pp and hh continuum
- Both terms yield negative contributions diverging near respective boundaries
- Only solutions for attraction indicated for one choice by horizontal dashed line
- Even true for very small coupling constant
- Stronger attraction -> complex eigenvalues



Can always get real eigenvalues by increasing gap!

Bound-pair states in nuclear matter

- Free space interaction generates deuteron bound state
- Scattering phase shifts indicate strong attraction in the medium
- Relevant eigenvalue problem (with gap in sp spectrum)



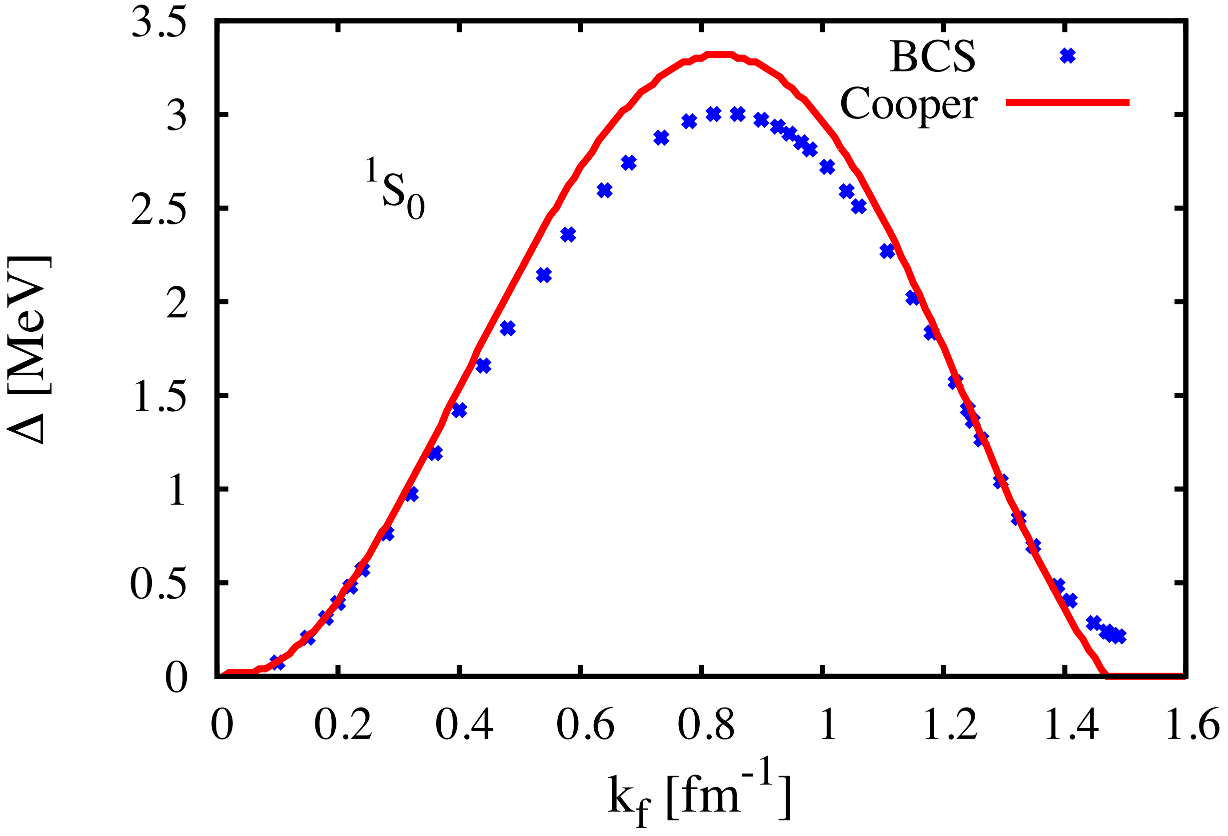
$$\begin{aligned}\psi_{BP}(k;(\ell S)JT) \quad &= \quad \frac{\theta(k-k_F)}{E_{BS}-2\varepsilon(k)}\;\frac{1}{2}\sum_{\ell'}\int\frac{dq\;q^2}{(2\pi)^3}\,\langle k\ell|\,V^{JST}\,|q\ell'\rangle\,\psi_{BP}(q;(\ell' S)JT)\\ &- \quad \frac{\theta(k_F-k)}{E_{BS}-2\varepsilon(k)}\;\frac{1}{2}\sum_{\ell'}\int\frac{dq\;q^2}{(2\pi)^3}\,\langle k\ell|\,V^{JST}\,|q\ell'\rangle\,\psi_{C}(q;(\ell' S)JT)\end{aligned}$$

- Gap required to avoid pairing instability sensitive function of density both for $^3S_1$-$^3D_1$ and $^1S_0$

Note zero density limit deuteron channel

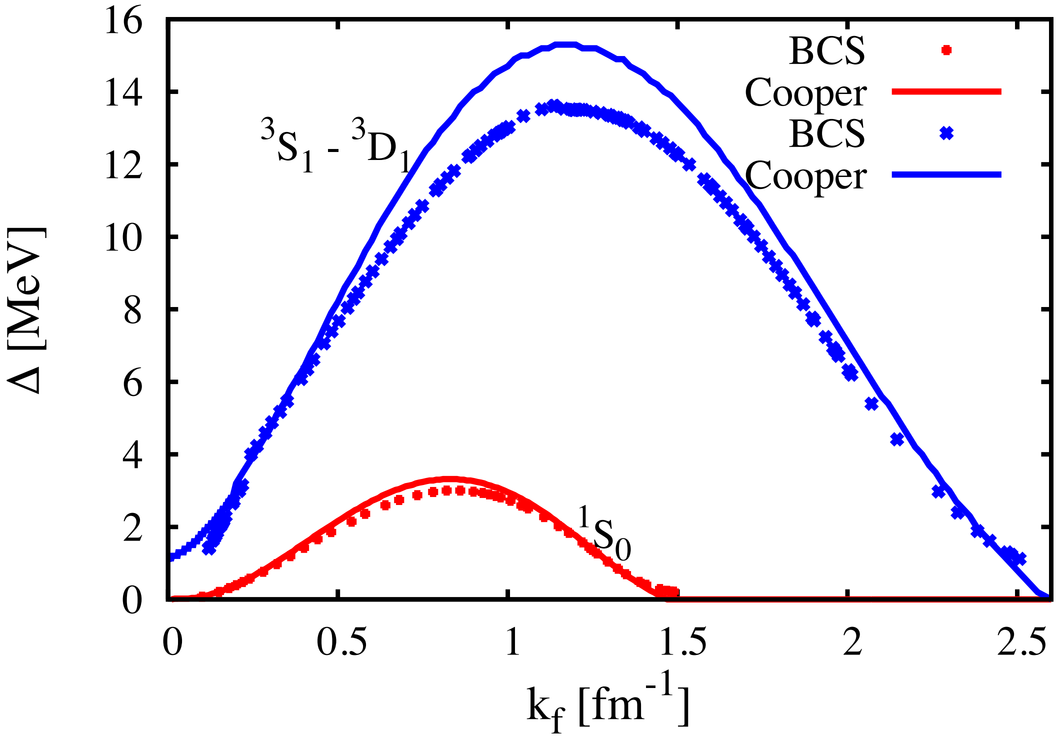
Compare with BCS gap calculation

- Already very close



BCS for $^3S_1$-$^3D_1$ in symmetric nuclear matter

- Puzzle



Mean-field particles

Early nineties: BCS gaps ~ 10 MeV

Alm et al. Z.Phys.A337,355 (1990)
Vonderfecht et al. PLB253,1 (1991)
Baldo et al. PLB283, 8 (1992)

Dressing nucleons is expected to reduce pairing strength as suggested by in-medium scattering

Bound-pair eigenvalues

- Gap required to high density
- Deuteron attraction greater than $^1S_0$
- Maximum sp gap ~ 15 MeV at $k_F = 1.2 fm^{-1}$
- Keep this gap for all densities to study eigenvalues
- Similarly for $^1S_0$ (> 3MeV gap)
- Also Cooper eigenvalue
- BCS approximately matches these results