Infinite nucleonic matter from coupled-cluster theory

Gaute Hagen (ORNL)

Lecture 1-2, TALENT school
Coupled-cluster computations of atomic nuclei

G. Hagen\textsuperscript{1,2}, T. Papenbrock\textsuperscript{2,1}, M. Hjorth-Jensen\textsuperscript{3,4}, and D. J. Dean\textsuperscript{1}

\textsuperscript{1}Physics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831 USA
\textsuperscript{2}Department of Physics and Astronomy, University of Tennessee, Knoxville, TN 37996, USA
\textsuperscript{3}National Superconducting Cyclotron Laboratory and Department of Physics and Astronomy, Michigan State University, East Lansing 48824 MI, USA
\textsuperscript{4}Department of Physics and Center of Mathematics for Applications, University of Oslo, N-0316 Oslo, Norway

E-mail: hageng@ornl.gov, tpapenbr@utk.edu, morten.hjorth-jensen@fys.uio.no, deandj@ornl.gov

Abstract. In the past decade, coupled-cluster theory has seen a renaissance in nuclear physics, with computations of neutron-rich and medium-mass nuclei. The method is efficient for nuclei with product-state references, and it describes many aspects of weakly bound and unbound nuclei. This report reviews the technical and conceptual developments of this method in nuclear physics, and the results of coupled-cluster calculations for nucleonic matter, and for exotic isotopes of helium, oxygen, calcium, and some of their neighbors.

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Outline

• Why study infinite nucleonic matter?
• Infinite nucleonic matter using periodic boundary conditions
• Setting up single-particle basis
• The Minnesota potential and computation of matrix elements in plane-wave basis
• Hartree-Fock for infinite matter using periodic boundary conditions
• Finite size effects and twist-averaging
• Going beyond mean-field
  – Many-body perturbation theory
  – Coupled-cluster in ladder approximation
  – Coupled-cluster doubles (CCD) approximation
• Recent applications of CCD to infinite nucleonic matter with chiral nucleon-nucleon and three-nucleon interactions
Why infinite nucleonic matter?

- Determines properties of supernova explosions and of neutron stars
- Equation of state links neutron stars to neutron radii in atomic nuclei
- The symmetry energy of nucleonic matter is related to the neutron skin


• The incompressibility is probed in giant dipole excitations in nuclei
• The nuclear matter saturation point from first principles is a challenge
Nucleonic matter from periodic boundary conditions

• Standard way to solve the infinite matter problem is to solve integral equations in relative center-of-mass frame using a partial wave expansion (e.g. Brueckner-Bethe-Goldstone theory)
  – Difficult to treat particle-hole correlations
  – Easy to extrapolate to thermodynamic limit

• Here we will focus on solving the infinite matter problem using periodic boundary conditions (PBC)
  – Straightforward to include all classes of correlations (e.g. particle-hole)
  – No partial wave expansion makes it easier to code
  – More difficult to extrapolate to thermodynamic limit
  – Finite size effects often non-negligible (can be minimized using twist-averaged periodic boundary conditions)
  – Due to momentum conservation there are no singles excitations
Periodic boundary conditions (PBC) in 3D

Schrodinger equation for plane waves in a cubic box

\[
-\frac{\hbar}{2m} \nabla^2 \phi(x) = \varepsilon \phi(x)
\]

Periodic boundary conditions gives:

\[
\phi(x, y, z) = \phi(x + L_x, y, z)
\]
\[
\phi(x, y, z) = \phi(x, y + L_y, z)
\]
\[
\phi(x, y, z) = \phi(x, y, z + L_z)
\]

Assume box of equal sides: \( L = L_x = L_y = L_z \)

Thies yields normalized plane wave solutions with discrete momenta:

\[
\phi_n(x) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}_n \cdot \mathbf{x}}
\]
\[
k_{n_i} = \frac{2\pi n_i}{L}
\]
\[
n_i = 0, \pm 1, \pm 2, \ldots
\]
Periodic boundary conditions in 3D

- The energy eigenvalues (ordered)
  \[ \varepsilon_n = \frac{\hbar^2}{2m} \left( k_{n_x}^2 + k_{n_y}^2 + k_{n_z}^2 \right) \]
  yield closed shell “Fermi” spheres:
  \[ n = 1, 7, 19, 27, 33, 57, \ldots \]
- Taking spin and isospin quantum numbers into account we have the single-particle basis
  \[ |\mathbf{k}_n, s_z, t_z\rangle = |\mathbf{k}\rangle \]
- Closed shells for nuclear matter is then:
  \[ n = 4, 28, 76, 108, 132, 228, \ldots \]
Setting up the basis for nuclear matter with PBC

- Specify number nucleons in the reference state (only for closed shells: 4,28,76,... etc)
- Specify density \( \rho = g_s k_F^3 / (6\pi^2) \) or Fermi-momentum \( k_F \) (\( g_s = 2 \) for neutron matter and \( g_s = 4 \) for nuclear matter)
- Compute the volume of the cubic box according to

\[
V = L^3 = A / \rho
\]

- From \( L \) we can compute the lattice spacing:

\[
\Delta k = 2\pi / L
\]

- Specify \( n_{\text{max}} \) for the basis
- Generate a Fortran/python/C++ code that set up a single particle basis for PBC

Me: Run Fortran example in terminal window
Setting up the basis for nuclear matter with PBC

• Student task:
  – Code an algorithm for setting up a single-particle basis for neutron/nuclear matter for given density/Fermi momentum, number of nucleons, and $n_{\text{max}}$
A simple central nucleon–nucleon potential:

The Minnesota potential: central Gaussian potential and diagonal in spin and isospin.

\[ V^c(r) = \frac{1}{2} \left( V_R + \frac{1}{2} (1 + P_{12}^\sigma) V_T + \frac{1}{2} (1 - P_{12}^\sigma) V_S \right) \left( 1 - P_{12}^\sigma P_{12}^\tau \right) \]

The spin and isospin exchange operators are given by:

\[ P_{12}^\sigma = \frac{1}{2} \left( 1 + (\sigma_1 \cdot \sigma_2) \right), \quad P_{12}^\tau = \frac{1}{2} \left( 1 + (\tau_1 \cdot \tau_2) \right) \]

The potential in r-space:

The parameters of the potential are given by:

\[
\begin{align*}
V_R &= V_{0R} \exp(-\kappa_R r_{12}^2) & V_{0R} &= 200 \text{ MeV}, \quad \kappa_R = 1.487 \text{ fm}^{-2} \\
V_T &= -V_{0T} \exp(-\kappa_T r_{12}^2) & V_{0T} &= 178 \text{ MeV}, \quad \kappa_T = 0.639 \text{ fm}^{-2} \\
V_S &= -V_{0S} \exp(-\kappa_S r_{12}^2) & V_{0S} &= 91.85 \text{ MeV}, \quad \kappa_S = 0.465 \text{ fm}^{-2}
\end{align*}
\]
Second quantized normal-ordered Hamiltonian:

\[ H = E_0 + H_N \]
\[ H_N = \sum_{pq} \langle k_p | f | k_q \rangle : a_p^\dagger a_q : \]
\[ + \frac{1}{4} \sum_{pqrs} \langle k_p k_q | v | k_r k_s \rangle : a_p^\dagger a_q^\dagger a_s a_r : \]

The vacuum energy:

\[ E_0 = \sum_i \langle k_i | t | k_i \rangle + \frac{1}{2} \sum_{i,j} \langle k_i k_j | V_{NN} | k_i k_j \rangle \]

The normal-ordered one-body part is given by:

\[ \langle k_p | f | k_q \rangle = \langle k_p | t | k_q \rangle + \sum k_p k_i | V_{NN} | k_q k_i \]

The normal-ordered two-body part is given by:

\[ \langle k_p k_q | v | k_r k_s \rangle = \langle k_p k_q | V_{NN} | k_r k_s \rangle \]

Note: all two-body matrix elements are here assumed to be anti-symmetric
Outline Lecture 2

- Python script for setting up single particle states
- The Minnesota potential and computation of matrix elements in plane-wave basis
- Hartree-Fock for infinite matter using periodic boundary conditions
- Finite size effects and twist-averaging
- Going beyond mean-field
  - Many-body perturbation theory
  - Coupled-cluster in ladder approximation
  - Coupled-cluster doubles (CCD) approximation
- Recent applications of CCD to infinite nucleonic matter with chiral nucleon-nucleon and three-nucleon interactions
Benchmark results for the pairing model

Can we understand the differences between ADC(2/3) and CCD?

What would be the leading order correction to CCD?
A simple central nucleon–nucleon potential:

The Minnesota potential: central Gaussian potential and diagonal in spin and isospin.

\[ V^c(r) = \frac{1}{2} \left( V_R + \frac{1}{2} (1 + P^\sigma_{12}) V_T + \frac{1}{2} (1 - P^\sigma_{12}) V_S \right) \left( 1 - P^\sigma_{12} P^\tau_{12} \right) \]

The spin and isospin exchange operators are given by:

\[ P^\sigma_{12} = \frac{1}{2} \left( 1 + (\sigma_1 \cdot \sigma_2) \right), \quad P^\tau_{12} = \frac{1}{2} \left( 1 + (\tau_1 \cdot \tau_2) \right) \]

The potential in r-space:

- The parameters of the potential are given by:
  
  \[ V_R = V_{0R} \exp(-\kappa_R r_{12}^2) \quad V_{0R} = 200 \text{ MeV}, \quad \kappa_R = 1.487 \text{ fm}^{-2} \]
  
  \[ V_T = -V_{0T} \exp(-\kappa_T r_{12}^2) \quad V_{0T} = 178 \text{ MeV}, \quad \kappa_T = 0.639 \text{ fm}^{-2} \]
  
  \[ V_S = -V_{0S} \exp(-\kappa_S r_{12}^2) \quad V_{0S} = 91.85 \text{ MeV}, \quad \kappa_S = 0.465 \text{ fm}^{-2} \]
Second quantized normal-ordered Hamiltonian:

Matrix elements of kinetic energy:
\[
\langle \mathbf{k}_p | t | \mathbf{k}_q \rangle = \frac{\hbar^2}{2m} \vec{k}_p^2 \delta_{pq}
\]

Matrix elements of a Gaussian nucleon-nucleon interaction are:
\[
V_R = V_0 \exp(-\alpha r_{12}^2)
\]
\[
\langle \mathbf{k}_p \mathbf{k}_q | v | \mathbf{k}_r \mathbf{k}_s \rangle = \frac{V_0}{L^3} \left( \frac{\pi}{\alpha} \right)^{3/2} \exp(-q^2/4\alpha) \delta_{\vec{k}_p + \vec{k}_q, \vec{k}_r + \vec{k}_s}
\]

\( q \) is the relative momentum transfer:
\( q = p - p' \)

Relative momenta are defined:
\[
p = \frac{1}{2}(\mathbf{k}_p - \mathbf{k}_q)
\]
\[
p' = \frac{1}{2}(\mathbf{k}_r - \mathbf{k}_s)
\]

Anti-symmetric matrix elements are then computed:
\[
\langle \mathbf{k}_p \mathbf{k}_q | v | \mathbf{k}_r \mathbf{k}_s \rangle_{AS} = \langle \mathbf{k}_p \mathbf{k}_q | v | \mathbf{k}_r \mathbf{k}_s \rangle - \langle \mathbf{k}_p \mathbf{k}_q | v | \mathbf{k}_s \mathbf{k}_r \rangle
\]

Note periodic boundary conditions ensure that momentum is conserved:
\[
\mathbf{k}_p + \mathbf{k}_q = \mathbf{k}_r + \mathbf{k}_s
\]
Matrix elements of the Minnesota potential

\[ \langle k_p k_q | v | k_r k_s \rangle = \]
\[ \frac{1}{2} \left( V_R + \frac{1}{2} V_T + \frac{1}{2} V_S \right) | k_r k_s \rangle + \]
\[ \frac{1}{4} (V_T - V_S) P_{12}^\sigma | k_r k_s \rangle - \]
\[ \frac{1}{2} \left( V_R + \frac{1}{2} V_T + \frac{1}{2} V_S \right) P_{12}^\sigma P_{12}^\tau | k_r k_s \rangle - \]
\[ \frac{1}{4} (V_T - V_S) P_{12}^\tau | k_r k_s \rangle \]

Matrix elements of the spin exchange operator are (and similarly for the isospin):

\[ \langle \sigma_p \sigma_q | P_{12}^\sigma | \sigma_r \sigma_s \rangle = \delta_{\sigma_p, \sigma_s} \delta_{\sigma_q, \sigma_r} \quad P_{12}^2 = 1 \]
Setting up the basis for nuclear matter with PBC

• Student task:
  – Derive a programmable expression for the Minnesota potential in a plane-wave basis for periodic boundary conditions:

\[
\langle \mathbf{k}_p \mathbf{k}_q | \mathbf{U} | \mathbf{k}_r \mathbf{k}_s \rangle
\]
Hartree–Fock for infinite matter

Hartree-Fock computed using PBC:

\[ E_0 = \sum_i \langle k_i \mid t \mid k_i \rangle + \frac{1}{2} \sum_{i,j} \langle k_i k_j \mid V_{NN} \mid k_i k_j \rangle \]

Hartree-Fock computed in relative center-of-mass coordinates for a central potential:

\[ \frac{E_0}{A} = \frac{3\hbar^2 k_F^2}{10m} + \frac{1}{(2\pi)^6} \frac{V}{2\rho} \sum_{t_{z_1} t_{z_2}} \sum_{s_{z_1} s_{z_2}} \int_0^{k_F} dk_1 \int_0^{k_F} dk_2 \langle k_{11} k_{22} \mid \nu \mid k_{11} k_{22} \rangle \]

Remember: \( |k_i\rangle = |\vec{k}_i, s_z, t_z\rangle \) and: \( A = \rho V \)

And matrix elements are anti-symmetric:

\[ \langle k_1 k_2 \mid \nu \mid k_1 k_2 \rangle = \langle k_1 k_2 \mid \nu \mid k_1 k_2 \rangle - \langle k_1 k_2 \mid \nu \mid k_2 k_1 \rangle \]
Hartree–Fock for infinite matter

Computing the six dimensional potential energy term for HF: can be computed “brute” force (but there is a simpler way).

\[
\frac{1}{(2\pi)^6} \frac{V}{2\rho} \sum_{t_{z_1} t_{z_2}} \sum_{s_{z_1} s_{z_2}} \int_0^{k_F} d\vec{k}_1 \int_0^{k_F} d\vec{k}_2 \langle \vec{k}_1 \vec{k}_2 | \nu | \vec{k}_1 \vec{k}_2 \rangle
\]

Note the interaction only depends on the momentum transfer squared \(q^2\)

\[
q^2 = p^2 - p'^2 - 2pp' \cos(\theta)
\]

\[
p = \frac{1}{2}(k_p - k_q)
\]

\[
p' = \frac{1}{2}(k_r - k_s)
\]

**Hint:** We can write the six dimensional integral as a three-dimensional integral by fixing one momenta along the z-axis and integrating over 3 angles (see notes on infinite matter at the github address of the course)
Hartree–Fock for infinite matter

Benchmark for Hartree-Fock energy in pure neutron matter for the Minnesota potential. PBC used 14 particles and $n_{\text{max}} = 3$. 
Hartree–Fock for infinite matter

Benchmark for Hartree-Fock energy in symmetric neutron matter for the Minnesota potential. PBC used 28 particles and $n_{\text{max}} = 3$.

Empirical saturation point

Does not saturate with Minnesota and HF
What could be missing?
Benchmark results for the Minnesota potential

Benchmark results for Hartree-Fock, MBPT2 and CCD energy in pure neutron matter (PNM) and symmetric nuclear matter (SNM) with the Minnesota potential. Here we used PBC with 14 particles for PNM and 28 particles for SNM with a model space size given by $n_{\text{max}} = 2$ (this gives 250 states for PNM and 500 states for SNM). Used the neutron mass $m c^2 = 939.565$ MeV for PNM and SNM.

<table>
<thead>
<tr>
<th>Type</th>
<th>$k_F$</th>
<th>$\rho$</th>
<th>$E0/A$</th>
<th>MBPT2/A</th>
<th>CCD/A</th>
</tr>
</thead>
<tbody>
<tr>
<td>PNM</td>
<td>1.33302</td>
<td>0.08</td>
<td>10.33372</td>
<td>9.70395</td>
<td>9.69991</td>
</tr>
<tr>
<td>PNM</td>
<td>1.43595</td>
<td>0.10</td>
<td>11.45846</td>
<td>10.74053</td>
<td>10.74307</td>
</tr>
<tr>
<td>PNM</td>
<td>1.52593</td>
<td>0.12</td>
<td>12.38970</td>
<td>11.59136</td>
<td>11.59903</td>
</tr>
<tr>
<td>SNM</td>
<td>1.33302</td>
<td>0.16</td>
<td>-23.33771</td>
<td>-26.34531</td>
<td>-28.26330</td>
</tr>
<tr>
<td>SNM</td>
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<td>-26.26943</td>
<td>-29.27810</td>
<td>-30.63909</td>
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<tr>
<td>SNM</td>
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<td>0.20</td>
<td>-29.19769</td>
<td>-32.20519</td>
<td>-33.22750</td>
</tr>
</tbody>
</table>

Comparison between AFDMC, CCD and CCD(PPHH Ladders)
Setting up the basis for nuclear matter with PBC

• Student task:
  – Code a program for computing the Hartree-Fock energy per nucleon in infinite neutron/nuclear matter for given density/Fermi momentum using the Minnesota nucleon-nucleon potential
  – Study how E/A changes with density/Fermi momentum
  – Study finite size effects by varying the particle number
  – Benchmark against Hartree-Fock computed in relative center-of-mass coordinates
Twist averaged periodic boundary conditions

- How to quantify error from shell oscillations/finite size effects?
- By increasing number of nucleons we don’t see a systematic and smooth convergence to the thermodynamic limit
- Comparing the kinetic energy computed with PBC and in the thermodynamic limit we can find an optimal number of particles
- Is there a more systematic way? Yes: Use twist-averaged PBC.

“Twist averaged” boundary conditions: average over all possible phases of Bloch waves. Exactly removes finite size effects for the free Fermi gas:

$$k_{n_i} = \frac{(2\pi n_i + \theta_i)}{L} \quad \psi_k(x + L) = e^{i\theta} \psi_k(x)$$


Discrete momenta are
**Twist averaged periodic boundary conditions**

- Finite size effects for kinetic energy in neutron matter for PBC and TAPBC
- With TAPBC we obtain a smooth/systematic convergence to thermodynamic limit

Choose a mesh (e.g. Gauss-Legendre) for the twist angles $\theta_{x,y,z}$ in the interval: $[0, \pi]$

Loop over mesh in $x,y,z$ and compute kinetic and hartree-fock energy

Very costly: for 10 points you need to compute the energy at one density $10^3$ times.

The final energy computed with TAPBC is

$$E_0^{TA} = E_0^{TA} + \left( \frac{1}{\pi^3} \right) \omega_{\theta_x} \omega_{\theta_y} \omega_{\theta_z} E_0(\theta_x, \theta_y, \theta_z)$$
Twist averaged periodic boundary conditions

Relative finite size corrections for the Hartree-Fock energy of pure neutron matter with the Minnesota potential

\[ \rho = 0.10 \text{ fm}^{-3}, \quad n_{\text{max}} = 3 \]
Quantifying errors from finite size effects

• Student task:
  – Implement twist-averaged periodic boundary conditions in pure neutron matter for the kinetic energy (possibly HF)
  – Study convergence to thermodynamic limit with number of closed shell neutrons and number meshpoints for the twist angles
Going beyond mean-field: MBPT2

- Correlations beyond mean-field are important in nuclear physics (strongly correlated many-body problem)
- First attempt is many-body perturbation theory at second order

\[ E_{\text{MBPT2}} = E_0 + \frac{1}{4} \sum_{ijab} \langle k_i k_j | v | k_a k_b \rangle \langle k_a k_b | t^0 | k_i k_j \rangle \]

Here \(a, b, c\), are orbitals above the Fermi level (unoccupied) and \(i, j, k\,...\) are orbitals below the Fermi level (occupied)

Hartree-Fock energy (1st order MBPT):

\[ E_0 = \sum_i \langle k_i | t | k_i \rangle + \frac{1}{2} \sum_{i,j} \langle k_i k_j | V_{NN} | k_i k_j \rangle \]

The \(t^0\) amplitudes are:

\[ \langle k_a k_b | t^0 | k_i k_j \rangle = \frac{\langle k_a k_b | v | k_i k_j \rangle}{f_{ii} + f_{jj} - f_{aa} - f_{bb}} \]

\[ f_{pq} = \langle k_p | f | k_q \rangle \]
Brief introduction to coupled-cluster diagrams

- Assign labels to particle and hole lines
- Matrix elements are in the order
- Each diagram gets a factor $\frac{1}{2}$ for equivalent lines
- Each diagram gets the phase $(-1)^{h+l}$
- Each diagram is anti-symmetrized in out/in-going lines

$$\langle \text{left}^{\text{out, right}^{\text{out}}} | v | \text{left}^{\text{in, right}^{\text{in}}} \rangle$$

$$= \frac{1}{2} \sum_{cd} \langle k_{a} k_{b} | v | k_{c} k_{d} \rangle \langle k_{c} k_{d} | t | k_{i} k_{j} \rangle$$

$$+ \frac{1}{2} \sum_{kl} \langle k_{a} k_{b} | t | k_{k} k_{l} \rangle \langle k_{k} k_{l} | v | k_{i} k_{j} \rangle$$
Going beyond mean-field with coupled-cluster method

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CCD(pphh)
Brueckner HF
Going beyond mean-field with coupled-cluster method
Going beyond mean-field: CCD_{ladd}

Note: For infinite nucleonic matter there are no singles excitations due to momentum conservation.

The CCD equations are given by:

\[ 0 = \langle \mathbf{k}_a \mathbf{k}_b | v | \mathbf{k}_i \mathbf{k}_j \rangle \]

\[ + P(ab) \sum_c \langle \mathbf{k}_b | f | \mathbf{k}_c \rangle \langle \mathbf{k}_a \mathbf{k}_c | t | \mathbf{k}_i \mathbf{k}_j \rangle \]

\[ - P(ij) \sum_k \langle \mathbf{k}_k | f | \mathbf{k}_j \rangle \langle \mathbf{k}_a \mathbf{k}_b | t | \mathbf{k}_i \mathbf{k}_k \rangle \]

\[ + \frac{1}{2} \sum_{cd} \langle \mathbf{k}_a \mathbf{k}_b | v | \mathbf{k}_c \mathbf{k}_d \rangle \langle \mathbf{k}_c \mathbf{k}_d | t | \mathbf{k}_i \mathbf{k}_j \rangle \]

\[ + \frac{1}{2} \sum_{kl} \langle \mathbf{k}_a \mathbf{k}_b | t | \mathbf{k}_k \mathbf{k}_l \rangle \langle \mathbf{k}_k \mathbf{k}_l | v | \mathbf{k}_i \mathbf{k}_j \rangle \]

The most costly term to evaluate in the CCD approximation. Computational cost is (naively):

\[ n_u^4 n_o^2 \]

But: Due to momentum conservation the cost is reduced to

\[ n_u^3 n_o \]
Going beyond mean-field: \( \text{CCD}_{\text{ladd}} \)

\[
D_{ij}^{ab} \langle k_a k_b | t | k_i k_j \rangle = \langle k_a k_b | v | k_i k_j \rangle \\
+ P(ab) \sum_{c \neq a} \langle k_b | f | k_c \rangle \langle k_a k_c | t | k_i k_j \rangle \\
- P(ij) \sum_{k \neq i} \langle k_k | f | k_j \rangle \langle k_a k_b | t | k_i k_k \rangle \\
+ \frac{1}{2} \sum_{cd} \langle k_a k_b | v | k_c k_d \rangle \langle k_c k_d | t | k_i k_j \rangle \\
+ \frac{1}{2} \sum_{kl} \langle k_a k_b | t | k_k k_l \rangle \langle k_k k_l | v | k_i k_j \rangle
\]

Solve non-linear CCD equations iteratively starting from the MBPT2 ansatz for \( t^0 \)

Convergence is improved by using a linear mixing between new and old amplitudes:

\[
t^{(i)} = \alpha t^{(i)} + (1 - \alpha) t^{(i-1)}
\]

Once the \( t \)-amplitudes are converged compute the CCD energy:

\[
E_{\text{CCD}} = E_0 + \frac{1}{4} \sum_{ijab} \langle k_i k_j | v | k_a k_b \rangle \langle k_a k_b | t | k_i k_j \rangle
\]
Solving the $\text{CCD}_{\text{ladd}}$ equations efficiently

\[ \frac{1}{2} \sum_{cd} \langle k_a k_b | v | k_c k_d \rangle \langle k_c k_d | t | k_i k_j \rangle \]

We can write the tensor contractions as a matrix-matrix multiplication:

\[ \frac{1}{2} \sum_{C} \langle A | v | C \rangle \langle C | t | I \rangle \]

Matrix-matrix multiplications can be efficiently solved by using linear algebra libraries such as LAPACK (Linear Algebra PACKage) and BLAS (Basic Linear Algebra Subprograms) routines.
Going beyond the mean-field

• Student task:
  – Implement MBPT2 for pure neutron matter and symmetric nuclear matter using PBC and the Minnesota potential
  – Code coupled-cluster for neutron matter and symmetric nuclear matter keeping only particle-particle and hole-hole ladder terms ($CCD_{ladd}$) and the Minnesota potential
Going beyond mean-field with coupled-cluster method

\[
0 = \begin{align*}
&\text{Diagram 1} \quad + \quad \text{Diagram 2} \\
&\quad + \quad \text{Diagram 3} \quad + \quad \text{Diagram 4}
\end{align*}
\]

CCD(pphh) Brueckner HF
Going beyond mean-field with coupled-cluster method

\[ 0 = \ \text{diagram} + \ \text{diagram} + \ \text{diagram} + \ \text{diagram} + \ \text{diagram} + \ \text{diagram} + \ \text{diagram} + \ \text{diagram} \]
Going beyond mean-field: CCD

The full CCD equations in factorized form:

\[
0 = \langle k_a k_b | v | k_i k_j \rangle \\
+ P(ab) \sum_c \langle k_b | \chi | k_c \rangle \langle k_a k_c | t | k_i k_j \rangle \\
- P(ij) \sum_k \langle k_k | \chi | k_j \rangle \langle k_a k_b | t | k_i k_k \rangle \\
+ \frac{1}{2} \sum_{cd} \langle k_a k_b | \chi | k_c k_d \rangle \langle k_c k_d | t | k_i k_j \rangle \\
+ \frac{1}{2} \sum_{kl} \langle k_a k_b | t | k_k k_l \rangle \langle k_k k_l | \chi | k_i k_j \rangle \\
+ P(ij) P(ab) \sum_{kc} \langle k_a k_c | t | k_i k_k \rangle \langle k_k k_b | \chi | k_c k_j \rangle
\]
Going beyond mean-field: CCD

One-body intermediates:

\[
\begin{align*}
\langle k_b | \chi | k_c \rangle &= \langle k_b | f | k_c \rangle \\
- \frac{1}{2} \sum_{kld} \langle k_b k_d | t | k_k k_l \rangle \langle k_k k_l | v | k_c k_d \rangle &+ \frac{1}{2} \sum_{cdl} \langle k_k k_l | v | k_c k_d \rangle \langle k_c k_d | t | k_k k_l \rangle
\end{align*}
\]

Two-body intermediates:

\[
\begin{align*}
\langle k_k k_l | \chi | k_i k_j \rangle &= \langle k_k k_l | v | k_i k_j \rangle \\
+ \frac{1}{2} \sum_{cd} \langle k_k k_l | v | k_c k_d \rangle \langle k_c k_d | t | k_i k_j \rangle + \frac{1}{2} \sum_{dl} \langle k_k k_l | v | k_c k_d \rangle \langle k_d k_b | t | k_l k_j \rangle
\end{align*}
\]
Why use intermediates?

\[ n_u^3 n_o^3 + n_u^2 n_o^2 + n_u^3 n_o^3 \quad n_u \gg n_o \]

\[ \sum_{k\text{c}} \langle k_a k_c | t | k_i k_k \rangle \langle k_k k_b | \chi | k_c k_j \rangle \]

\[ = \sum_{k\text{c}} \langle k_a k_c | t | k_i k_k \rangle \langle k_k k_b | v | k_c k_j \rangle \]

\[ + \frac{1}{2} \sum_{k\text{cld}} \langle k_a k_c | t | k_i k_k \rangle \langle k_k k_l | v | k_c k_d \rangle \langle k_b k_d | t | l_i k_j \rangle \]

\[ n_u^3 n_o^3 + n_u^4 n_o^4 \]
Going beyond the mean-field

• Student task:
  – Derive the CCD equations from the CCD diagrams using the diagrams rules
  – Write the CCD equations in a factorized form using intermediates
  – Implement CCD for symmetric nuclear matter and pure neutron matter
  – Do a comparison between HF, MBPT2, \( \text{CCD}_{\text{ladd}} \) and CCD in pure neutron matter and nuclear matter around saturation density
### Nuclear forces from chiral effective field theory

[Weinberg; van Kolck; Epelbaum et al.; Entem & Machleidt; …]

<table>
<thead>
<tr>
<th></th>
<th>NN</th>
<th>3N</th>
<th>4N</th>
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<tr>
<td><strong>LO</strong> $\mathcal{O}(\frac{Q_0}{A_0})$</td>
<td><img src="LO_diagram.png" alt="Diagram" /></td>
<td><img src="3N_diagram.png" alt="Diagram" /></td>
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<tr>
<td><strong>NLO</strong> $\mathcal{O}(\frac{Q^2}{A^2})$</td>
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<tr>
<td><strong>N^2LO</strong> $\mathcal{O}(\frac{Q^3}{A^3})$</td>
<td><img src="N2LO_diagram.png" alt="Diagram" /></td>
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<td><img src="4N_diagram.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

- developing higher orders and higher rank (3NF, 4NF) [Epelbaum 2006; Bernard et al 2007; Krebs et al 2012; Hebeler et al 2015; …]
- implemented in continuum and on lattice [Borasoy et al 2007]
- local / non-local formulations [Gezerlis et al 2013]
- propagation of uncertainties on horizon [Navarro Perez 2014]
- different optimization protocols [Ekström et al 2013]

Much improved understanding and handling via renormalization group transformations [Bogner et al 2003; Bogner et al 2007]
Benchmark calculations of neutron matter

Comparison with continuum coupled-cluster method [Baardsen et al., PRC (2013)]

Comparison with auxiliary field diffusion Monte Carlo and Minnesota potential
Role of particle–hole excitations in nucleonic matter

- Particle-hole and non-linear terms in CCD are small in pure neutron matter.
- MBPT2/CCD\textsubscript{ladd}/CCD results agree within 500keV/A. Indicates that PNM is perturbative.
- Particle-hole and non-linear terms play a larger role in symmetric nuclear matter.
- CCD\textsubscript{ladd} and CCD results differ by up to 1.5MeV/A around saturation density.
Three nucleon force (3NF) and regulator dependence


**Nonlocal form of 3NF** [Epelbaum et al. PRC (2002)]: Cutoff is in Jacobi momenta
\( \Lambda = 500 \text{ MeV} \): \( c_D = -2, c_E = -0.791 \) (from \( A=3 \) binding energies)

**Local form of 3NF** [Navratil, Few Body Syst. (2007)]: Cutoff is in the momentum transfer
\( \Lambda = 500 \text{ MeV} \): \( c_D = -0.39, c_E = -0.389 \) (from \( A=3,4 \) binding and \( ^3\text{H} \tau_{1/2} \) (Gazit, Navratil, & Quaglioni)
\( \Lambda = 400 \text{ MeV} \): \( c_D = -0.39, c_E = -0.27 \) (adjusted to \( ^4\text{He} \))
Neutron matter is perturbative (small differences between MBPT2 and coupled clusters). 3NFs act repulsively in neutron matter and $\text{NNLO}^{\text{opt}}$. Error bands from variation of cutoffs and level of sophistication in treating 3NFs.

For non-local regulators only $c_1$ and $c_3$ contribute.
For local regulators all terms contribute.
Symmetric nuclear matter

Nuclear matter is not perturbative (larger differences between MBPT2 and coupled clusters)
3NFs act repulsively in nuclear matter and \( \text{NNLO}_{\text{opt}} \)

\( \text{Regularization scheme dependence of 3NF; sensitivity to sophistication in treatment of 3NFs} \)
5% error bands for saturation $k_f$, $E/N$ and binding energy of $^3$H

$\bigcirc$ Variation of $c_D$ and $c_E$ not sufficient to simultaneously bind light nuclei and nuclear matter

Cutoff dependent fraction of residual 3NF contribution to MBPT2 energy per particle in SNM around saturation density.
Local regulators converge slower than non-local regulators
Our solution: simultaneous optimization of NN and 3NFs with input from selected nuclei up to $A \sim 25$ (NNLO$_{\text{sat}}$). A. Ekström et al, Phys. Rev. C 91, 051301(R) (2015)