Lecture 1. Nuclear Matter. Brueckner-Hartree-Fock

- The many-body problem in nuclear matter
- The NN interaction and the need of sophisticated many-body mehods
- T--matrix and the summation of ladder diagrams
- G-matrix the summation of ladder particle-particle diagrams in the medium.
- How to calculate the self-energy?
- Use of effective interactions in the HF approximation

A great effort is being devoted to study the properties of asymmetric nuclear systems both from experimental and theoretical points of view.

"ab initio" calculations could be a safe way to study these systems. However, this procedure could mean different things …

1. Choose degrees of freedom: nucleons
2. Choose interaction: Realistic phase-shift equivalent two-body potential (CDBONN, Av18, N3LO).
3. Select three-body force

With these ingredients we build a non-relativistic Hamiltonian ===> Many-body Schrodinger equation. To solve this equation (ground or excited states) one needs a sophisticated many-body machinery.

We need as good as posible many-body theories to eliminate uncertainties!

Remember:

Nucleon-nucleon interaction is not uniquely defined.
Complicated channel structure. Tensor component in the NN interaction.
Already the deuteron is complicated.

An example …

Argonne v18

$$v(NN) = v^{EM}(NN) + v^{\pi}(NN) + v^{R}(NN)$$

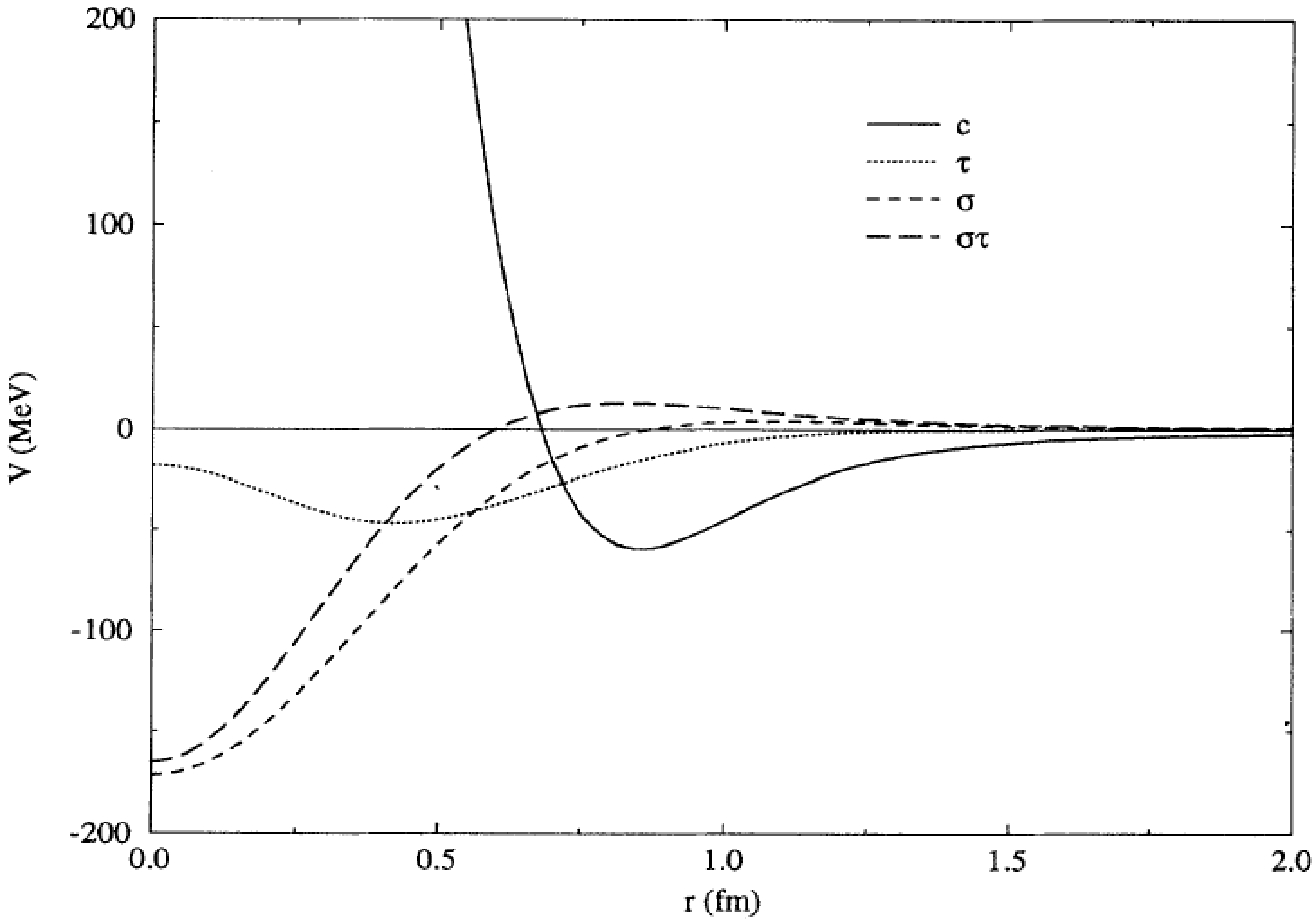
is the sum of 18 operators that respect some symmetries. components 15-18 violate charge indepedence.

$$v_{ij} = \sum_{p=1,18} v_p(r_{ij}) O_{ij}^p$$

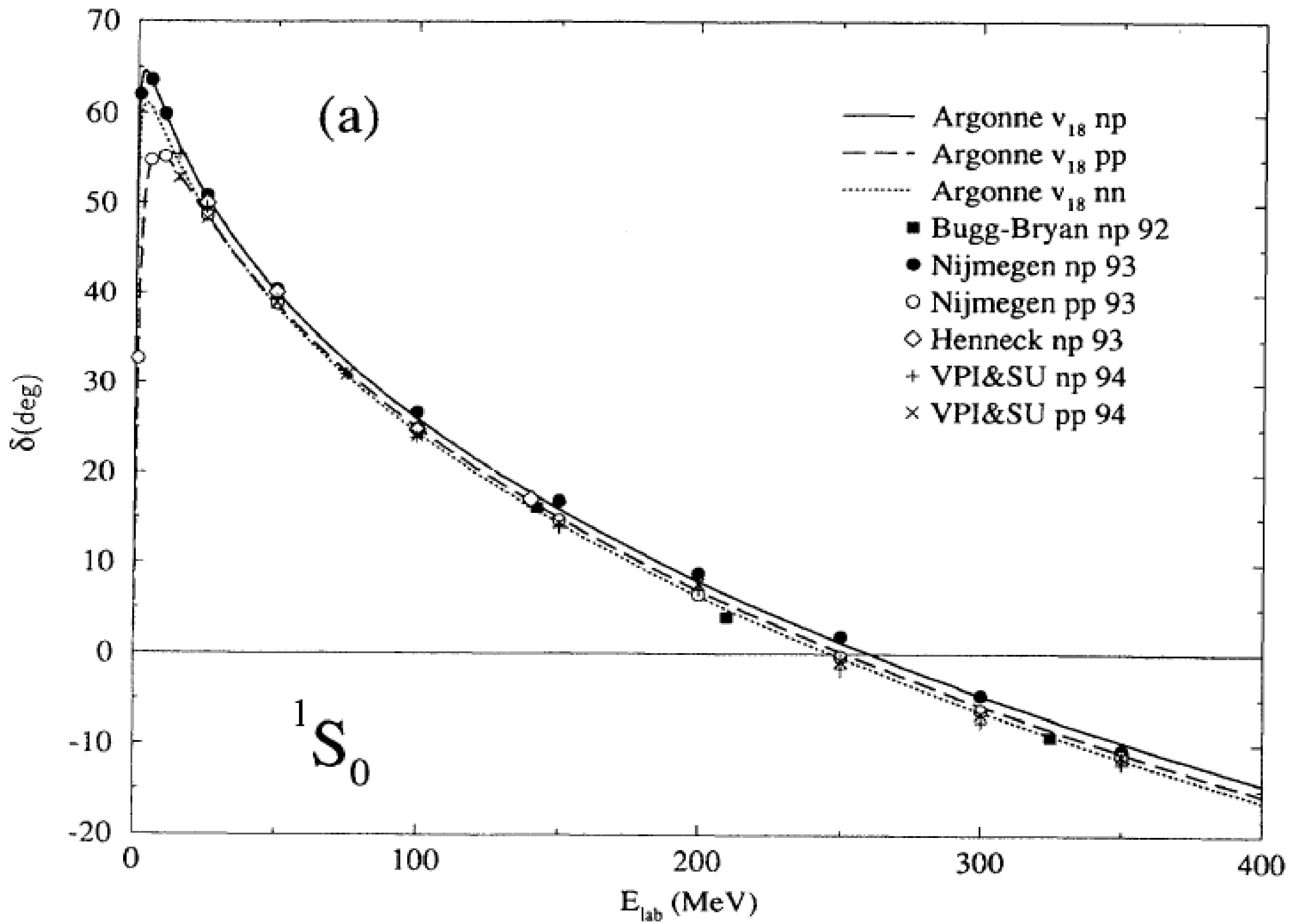
$$O_{ij}^{p=1,14} = 1, \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j,\ \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j, (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j),\ S_{ij}, S_{ij}(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j),\ \mathbf{L}\cdot\mathbf{S}, \mathbf{L}\cdot\mathbf{S}(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j),$$
$$L^2, L^2(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j),\ L^2(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j), L^2(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j),\ (\mathbf{L}\cdot\mathbf{S})^2, (\mathbf{L}\cdot\mathbf{S})^2(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$$

$$O_{ij}^{p=15,18} = T_{ij},\ (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)T_{ij},\ S_{ij}T_{ij},\ (\tau_{zi} + \tau_{zj})$$

Central, isospin, spin, and spin-isopin components.
The repulsive short-range of the central part has a peak value of 2031 MeV at r=0.



Phase shifts in the 1S0 channel.



Perturbative methods: Due to the short-range structure of a realistic potential ➔ Order by order perturbation theory is not posible ➔ infinite partial summations.

Diagrammatic notation is useful.

Brueckner-Hartree-Fock. G-matrix

Main issue the energy of the ground state

Self- Consistent Green's function (SCGF)

Single-particle properties and also the binding energy.

A simple option:

Variational methods as FHNC or VMC

$$\Psi(1,...,N) = F(1,..,N)\phi(1,...,N)$$

$$F(1,...N) = \prod_{i<j} f^{(2)}(ij)$$

$$E = \frac{\langle \Psi \mid H \mid \Psi \rangle}{\langle \Psi \mid \Psi \rangle}$$

Quantum Monte Carlo: GFMC and AFDMC. Simulation box with a finite number of particles. Special method for sampling the operatorial correlations.

The microscopic study of nuclear systems requires a rigorous treatment of the nucleon-nucleon (NN) correlations.

- Strong short range repulsion and tensor components in realistic interactions, to fit NN scattering data, produce important modifications of the nuclear wave function.
- Simple Hartree-Fock for nuclear matter at the empirical saturation density using such realistic NN interactions provides positive energies rather than the empirical -16 MeV per nucleon.
- The effects of correlations appear also in the single-particle properties:
- Partial occupation of the single particle states which would be fully occupied in a mean field description and a wide distribution in energy of the single-particle strength. The departure of n(k) from the step function (in a uniform system) gives a measure of the importance of correlations.

Symmetric Nuclear matter: Is a uniform system of equal number of structureless neutrons and protons which interact via a non-relativistic nucleon-nucleon potential, which are required to reproduce properties of a two-nucleon system. The Coulomb Interaction is turned off.

The zero-order approximation

The free (non-interacting) fermion gas

* Interparticles interactions are neglected.

The single-particle states used in the construction of the Fock basis are plane-wave states associated to the kinetic energy operator: $\frac{\vec{p}^2}{2m}$

As we will consider N nucleons inside a cubic box of volume $V = L^3$, we will use box normalization and periodic boundary conditions

Each non-interacting nucleon is characterized by a normalized momentum eigenstate

$$\boxed{\langle \vec{r} \,|\, \vec{k} \rangle = \frac{1}{\sqrt{V}} e^{i\vec{k}\vec{r}}}$$

where $\vec{k} = \left(n_x \frac{2\pi}{L}, n_y \frac{2\pi}{L}, n_z \frac{2\pi}{L} \right)$ $n_x, n_y, n_z = 0, \pm 1, \pm 2, \dots$

These states are orthonormal:

$$\langle \vec{k} \,|\, \vec{k}' \rangle = \delta_{\vec{k},\vec{k}'}$$ Kronecker delta

Boundary conditions allow only discrete values of the momentum.

* The Pauli principle allows only a fixed number of fermions in each single-particle momentum eigenstate, depending on the spin/isospin degeneracy of the system.

$\Rightarrow$ The ground state is obtained by filling the momentum allowed states up to a maximum value $\Rightarrow$ the Fermi momentum k_F

$$\boxed{|\phi_0\rangle = \prod_{|\vec{k}|<k_F} a^{+}_{\vec{k}\sigma}|0\rangle}$$

σ accounts for the spin/isospin quantum numbers

* This single particle basis can also be used in the presence of interparticle interactions!

* At the end, the volume and the number of particles are let to go to infinity, $V\to\infty$, $N\to\infty$ such that $\rho = \frac{N}{V}$ is kept fixe (Thermodynamic limit).

Relation between ρ and k_F

$$N = \langle\phi_0|\hat{N}|\phi_0\rangle = \sum_{\vec{k}\mu}\langle\phi_0|a^{+}_{\vec{k}\mu}a_{k\mu}|\phi_0\rangle = \sum_{\vec{k}\mu}\theta(k_F-k)$$

For large V, $\sum_{\vec{k}} = \frac{V}{(2\pi)^3}\int d^3k$

$$N = \frac{V}{(2\pi)^3}\sum_{\mu}\int d^3k\ \theta(k_F-k) = \frac{\nu V}{6\pi^2}k_F^3 \Rightarrow \boxed{\rho = \frac{\nu}{6\pi^2}k_F^3}$$

Relation between the density and the highest occupied k

$\nu = 4$ nuclear matter

$\nu = 2$ neutron matter

* The kinetic energy of these N nucleons:

$$H_0 = \sum_{\vec{k}\mu}' \frac{\hbar^2 k^2}{2m} \longrightarrow \hat{T} = \sum_{\vec{k}\mu}' \frac{\hbar^2 k^2}{2m} a^{+}_{\vec{k}\mu} a_{\vec{k}\mu}$$

where we have taken into account that the kinetic energy is diagonal in the momentum basis.

* Actually, $|\phi_0\rangle$ is an eigenstate of $\hat{T}$

$$\hat{T}\,|\phi_0\rangle = \underbrace{\left(\sum_{\substack{|\vec{k}|<k_F \\ \mu}}' \frac{\hbar^2 k^2}{2m} \right)}_{E_0} |\phi_0\rangle$$

The eigenenergie is the sum of the kinetic energies of the occupied states.

$$E_0 = \sum_{|\vec{k}|<k_F,\mu} \frac{\hbar^2 k^2}{2m} = \frac{V}{(2\pi)^3} \sum_\mu \int d^3k \, \frac{\hbar^2 k^2}{2m} \, \Theta(k_F - k)$$

$$= V \frac{\nu}{(2\pi)^3} 4\pi \frac{\hbar^2}{2m} \frac{1}{5} k_F^5 = N \cdot \frac{3}{5} \frac{\hbar^2 k_F^2}{2m}$$

$$N = \frac{\nu V}{(2\pi)^3} \frac{4}{3} \pi k_F^3$$

and the kinetic energy per particle:

$$e = \frac{E_0}{N} = \frac{3}{5} \frac{\hbar^2 k_F^2}{2m}$$

which in terms of the density

$$k_F = \left(\frac{6\pi^2 \rho}{\nu}\right)^{1/3}$$

$$\Rightarrow \boxed{e = \frac{3}{5} \frac{\hbar^2}{2m} \left(\frac{6\pi^2}{\nu}\right)^{2/3} \rho^{2/3}}$$

For nuclear matter, $\nu = 4$

$$\frac{\hbar^2}{2m} = \underline{20.74 \text{ MeV} \cdot \text{fm}^2} \quad \Rightarrow \quad e = 75.03\, \rho^{2/3} \text{ MeV}$$

* The energy increases monotonically with the density

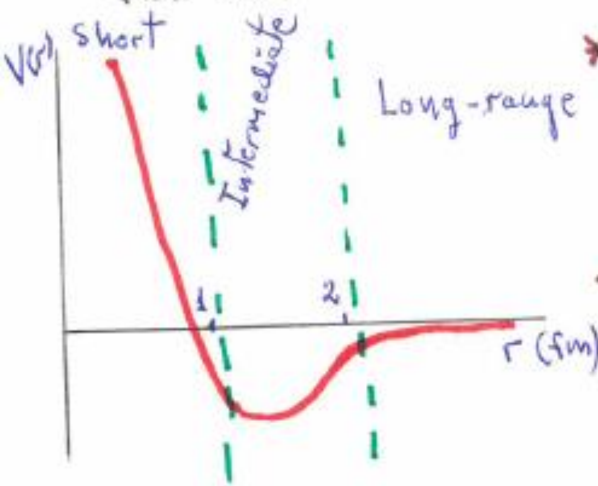
If nuclear matter must be stable at the so called saturation density $\rho = \rho_0 \simeq 0.17 \text{ fm}^{-3}$ we need an attractive potential energy around this density such that the energy has a minimum at this density.

$$k_F = \left(\frac{6\pi^2 \rho}{\nu}\right)^{1/3} \simeq 1.36 \text{ fm}^{-1}$$

For, $\rho_0 \simeq 0.17 \text{ fm}^{-3} \Rightarrow \quad e = \frac{3}{5} \frac{\hbar^2 k_F^2}{2m} \simeq 23.02 \text{ MeV}$

We expect that the nuclear matter is self-bound at this density (saturation density) with a binding energy per nucleon $\Rightarrow \dfrac{B}{N} \simeq -16 \text{ MeV}$

We need to add some attraction to the free Fermi gas ! ⇒ Consider the nucleon-nucleon interaction.



* At large distance $r \geq 2$ fm the interaction is attractive with an exponential tail

* At intermediate distance, $0.4 < r < 2$ fm, is also attractive (in average)

* At short distances $r \leq 0.4$ fm a strong repulsion (core) is present.

Assuming that only two-body interaction is present ⇒ The Hamiltonian can be written as:

$$\hat{H} = \hat{H}_0 + \hat{H}_1 = \sum_{\{k\}} \frac{\hbar^2 \vec{k}^2}{2m} a_k^+ a_k + \frac{1}{2} \sum_{\{k_i\}} \langle k_1 k_2 | V | k_3 k_4 \rangle a_{k_1}^+ a_{k_2}^+ a_{k_4} a_{k_3}$$

System of A fermions described by

$$H = \sum_{i=1}^{A} T_i + \sum_{i<j}^{A} V_{ij}$$

Ground State $\Rightarrow$ $H|\Psi\rangle = E|\Psi\rangle$

But unsolvable

$\Downarrow$

$$H = \boxed{\sum_{i=1}^{A}(T_i + U_i)} + \boxed{\sum_{i<j}^{A} V_{ij} - \sum_{i=1}^{A} U_i}$$

unperturbed

perturbation

$$= H_0 + H_1$$

$\Downarrow$

$$E = E_0 + \Delta E$$

$$H_0|\Phi_0\rangle = E_0|\Phi_0\rangle$$

$\Delta E \rightarrow$ perturbation theory

Perturbation theory gives a formal expansion for $\Delta \mathrm{E}$

The operator P projects off the ground state and ensures that the ground state Does not take place as an intermediate state.

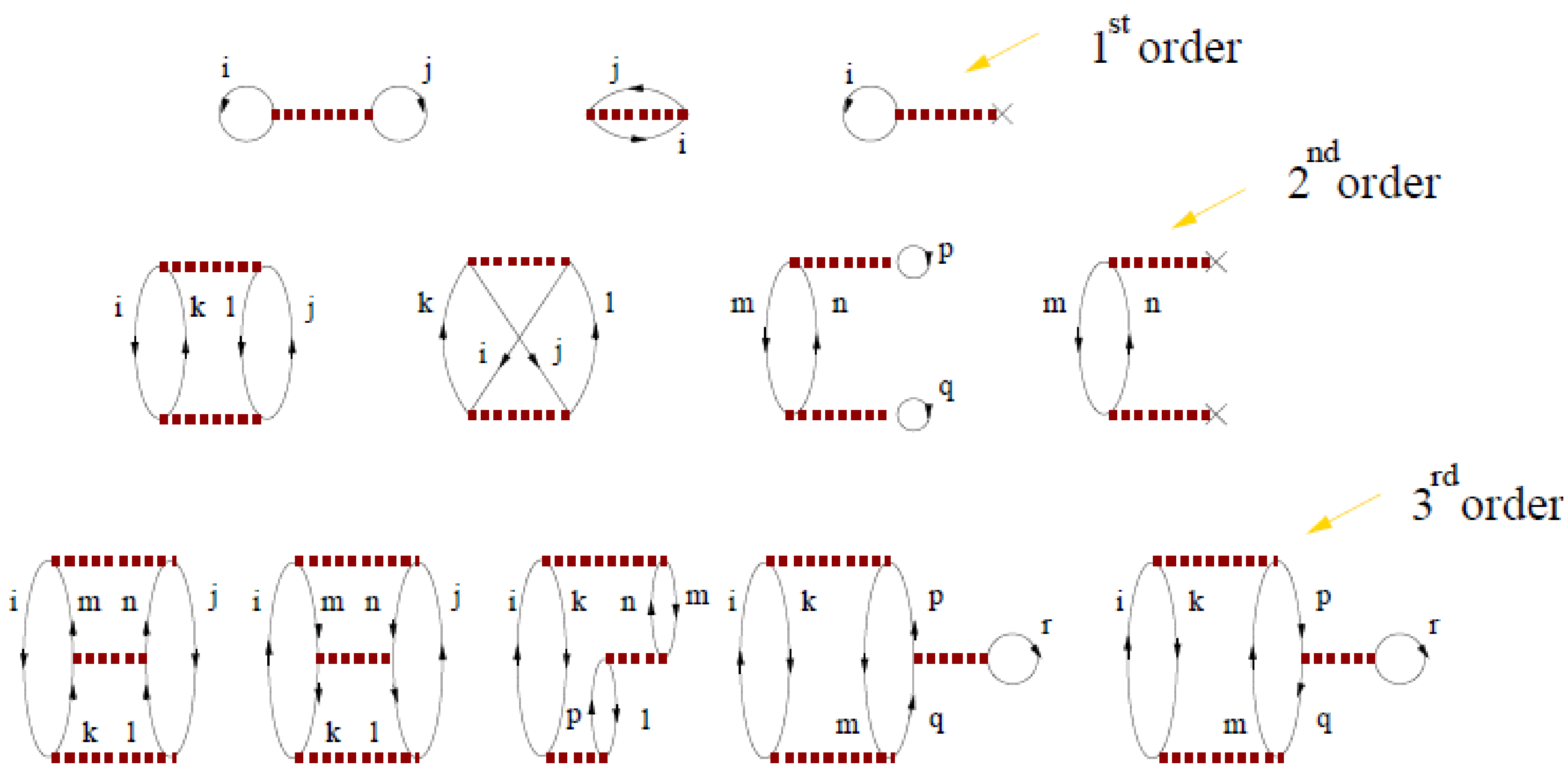
$$\Delta E = \langle\Phi_0|H_1|\Phi_0\rangle + \langle\Phi_0|H_1 \frac{1-|\Phi_0\rangle\langle\Phi_0|}{E_0 - H_0} H_1|\Phi_0\rangle$$

$$+\langle\Phi_0|H_1 \frac{1-|\Phi_0\rangle\langle\Phi_0|}{E_0 - H_0} H_1 \frac{1-|\Phi_0\rangle\langle\Phi_0|}{E_0 - H_0} H_1|\Phi_0\rangle$$

$$-\langle\Phi_0|H_1|\Phi_0\rangle\langle\Phi_0|H_1 \frac{1-|\Phi_0\rangle\langle\Phi_0|}{(E_0 - H_0)^2} H_1|\Phi_0\rangle + \cdots$$

$$= \langle\Phi_0|H_1 \sum_{n=0}^{\infty} \left[\frac{1-|\Phi_0\rangle\langle\Phi_0|}{E_0 - H_0} H_1\right]^n |\Phi_0\rangle_l$$

Goldstone expansion



First Order correction

$$V = \sum_{i<j}' V_{ij} = \frac{1}{2} \sum_{i \neq j}' V_{ij}$$

$$\frac{\langle \Phi_{FS} | \sum_{i<j}' v(r_{ij}) | \Phi_{FS} \rangle}{N} = \frac{1}{2} \frac{1}{N} \sum_{\alpha \beta}' \langle \alpha \beta | v_{12} | \alpha \beta - \beta \alpha \rangle$$

$$|\alpha\rangle = |k\rangle \, |m_s\rangle \, |m_t\rangle$$

$$\langle \vec{r} | \vec{k} \rangle = \frac{1}{\sqrt{\Omega}} e^{i \vec{k} \vec{r}}$$ Normalized to volume (box)

* The single particle states are characterized by the momentum, and the third component of the spin. The third component of the isospin indicates if the nucleon is a proton or a neutron.

* The sum over the states, reduces to a sum over $\vec{k}$, m_s and m_ℓ.

* To perform the sum over $\vec{k} \Rightarrow \sum_{\vec{k}} = \frac{\Omega}{(2\pi)^3} \int d^3k$

in the case of Φ_{FS}, the integral is limited inside the Fermi sphere $\Rightarrow$

$$\boxed{\sum_{\vec{k}} = \frac{\Omega}{(2\pi)^3} \int_{|\vec{k}| \leq k_F} d^3k}$$

First order correction for a simple central potential

$$\frac{\langle V \rangle}{N} = \frac{1}{2}\frac{1}{N}\sum_{\{k,k'\}}' \langle \vec{k}\, m_s\, m_t,\ \vec{k}'\, m_s'\, m_t' \,|\, \vartheta(r_{12}) \,|\, \vec{k}\, m_s\, m_t,\ \vec{k}'\, m_s'\, m_t' - \vec{k}'\, m_s'\, m_t',\ \vec{k}\, m_s\, m_t \rangle$$

$$= \frac{1}{2}\frac{1}{N}\frac{\Omega^2}{(2\pi)^6}\int_{k\le k_F} d^3k \int_{k'\le k_F} d^3k' \sum_{m_s m_s'}' \sum_{m_t m_t'}'$$

$$\Big[\int d^3r_1 \int d^3r_2 \frac{1}{\sqrt{\Omega}} e^{-i\vec{k}\vec{r}_1} \frac{1}{\sqrt{\Omega}} e^{-i\vec{k}'\vec{r}_2}\, \vartheta(r_{12})\, \frac{1}{\sqrt{\Omega}} e^{-i\vec{k}\vec{r}_1} \frac{1}{\sqrt{\Omega}} e^{-i\vec{k}'\vec{r}_2}$$

$$\langle m_s\, m_s' \,|\, m_s\, m_s' \rangle \langle m_t\, m_t' \,|\, m_t\, m_t' \rangle$$

$$- \int d^3r_1 \int d^3r_2 \frac{1}{\sqrt{\Omega}} e^{-i\vec{k}\vec{r}_1} \frac{1}{\sqrt{\Omega}} e^{-i\vec{k}'\vec{r}_2}\, \vartheta(r_{12})\, \frac{1}{\sqrt{\Omega}} e^{+i\vec{k}'\vec{r}_1} \frac{1}{\sqrt{\Omega}} e^{i\vec{k}\vec{r}_2}$$

$$\langle m_s\, m_s' \,|\, m_s'\, m_s \rangle \langle m_t\, m_t' \,|\, m_t'\, m_t \rangle \Big]$$

$$\langle m_s\, m_s' \mid m_s\, m_s' \rangle = 1$$

$$\langle m_s\, m_s' \mid m_s'\, m_s \rangle = \delta_{m_s m_s'}$$ Trace of the identity

$$\sum_{m_s m_s'} \langle m_s\, m_s' \mid m_s\, m_s' \rangle = \sum 1 = \mathrm{Tr}\,(I) = \nu_s^2$$

$$\sum_{m_s m_s'} \langle m_s\, m_s' \mid m_s'\, m_s \rangle = \sum_{m_s m_s'} \langle m_s\, m_s' \mid P_\sigma \mid m_s\, m_s' \rangle$$

$$= \mathrm{Tr}\,(P_\sigma) = \nu_s$$ Trace of the exchange operator

The same for the isospin.

$$= \frac{1}{2}\,\frac{1}{N}\,\frac{\Omega^2}{(2\pi)^6}\,\frac{1}{\Omega^2} \int_{k \le k_F} d^3k \int_{k' \le k_F} d^3k' \int d^3r_1 \int d^3r_2 \; \vartheta(r_{12}) \left(\nu_s^2 \nu_t^2 - \nu_s \nu_t \, e^{i(\bar{k}-\bar{k}')(\bar{r}_2-\bar{r}_1)} \right)$$

Performing first the integral over momenta, which is independent of the potential

$$\frac{\langle V \rangle}{N} = \frac{1}{2}\,\frac{1}{N}\,\frac{1}{(2\pi)^6}\,\nu_s^2 \nu_c^2 \int d^3r_1 \int d^3r_2 \; v(r_{12}) \int_{k\le k_F} d^3k \int_{k'\le k_F} d^3k' \left(1 - \frac{1}{\nu_s \nu_c}\, e^{i\bar{k}(\bar{r}_2-\bar{r}_1)}\, e^{-i\bar{k}'(\bar{r}_2-\bar{r}_1)}\right)$$

$$\int_0^{k_F} d^3k = \frac{4}{3}\pi k_F^3 = (2\pi)^3 \frac{\rho}{\nu_s \nu_c}$$

$$\downarrow$$

$$\nu_s \nu_c \sum_k 1 = N = \nu_s \nu_c \frac{\Omega}{(2\pi)^3} \int_{k\le k_F} d^3k = \nu_s \nu_c \frac{\Omega}{(2\pi)^3}\,\frac{4}{3}\pi k_F^3$$

$$\int_{k\le k_F} d^3k \; e^{i\bar{k}\bar{r}} = \int_{k\le k_F} dk \; k^2 \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta) \left(\cos kr(\cos\theta) + i \sin kr(\cos\theta)\right)$$

$$\int_{-1}^{1} dx \left(\cos krx + i \sin krx\right) = \int_{-1}^{1} dx \cos krx$$

$$= \left.\frac{\sin krx}{kr}\right]_{-1}^{1} = \frac{2\sin kr}{kr}$$

$$\int_{k<k_F} d^3k \; e^{i\bar{k}\bar{r}} = 2\pi \int_0^{k_F} dk \; k^2 \, \frac{2\sin kr}{kr} = \frac{4\pi}{r}\left[\frac{\sin k_F r}{r^2} - \frac{k_F \cos k_F r}{r}\right]$$

Slater function

$$\ell(k_F r) = \frac{3\, j_1(k_F r)}{k_F r} \qquad j_1(k_F r) = \frac{\sin(k_F r)}{(k_F r)^2} - \frac{\cos(k_F r)}{k_F r}$$

$$\int_{k<k_F} d^3k \; e^{i\bar{k}\bar{r}} = \frac{4\pi k_F^3}{k_F r}\left[\frac{\sin k_F r}{(k_F r)^2} - \frac{\cos k_F r}{k_F r}\right] = 4\pi k_F^3 \frac{j_1(k_F r)}{k_F r}$$

$$= \frac{6\pi^2 \rho}{\nu_s \nu_\ell}\, 4\pi \frac{j_1(k_F r)}{k_F r} = (2\pi)^3 \frac{\rho}{\nu_s \nu_\ell} \frac{3\, j_1(k_F r)}{k_F r} = (2\pi)^3 \frac{\rho}{\nu}\, \ell(k_F r)$$

$$\ell(k_F r) = \frac{\nu_s \nu_\ell}{(2\pi)^3 \rho} \int_{k\le k_F} d^3k \; e^{i\bar{k}\bar{r}}$$

$$\frac{\langle V\rangle}{N} = \frac{1}{2}\,\frac{1}{N}\,\frac{1}{(2\pi)^6}\,\nu_s^2\nu_t^2 \int d^3r_1\, d^3r_2\;\;\vartheta(r_{12}) \int d^3k\, d^3k' \left(1 - \frac{1}{\nu_s\nu_t}\, e^{i\bar{k}(\bar{r}_2-\bar{r}_1)} \cdot e^{-i\bar{k}'(\bar{r}_2-\bar{r}_1)}\right)$$

$$= \frac{1}{2}\,\frac{1}{N}\,\frac{\nu_s^2\nu_t^2}{(2\pi)^6}\left[(2\pi)^3\frac{\rho}{\nu_s\nu_t}\right]^2 \iint d^3r_1\, d^3r_2\;\;\vartheta(r_{12}) \left(1 - \frac{1}{\nu_s\nu_t}\,\ell^2(k_F r)\right)$$

$$= \frac{1}{2}\,\frac{1}{N}\,\rho^2 \int d^3r_1\;\; d^3r_{12}\;\;\vartheta(r_{12}) \left(1 - \frac{1}{\nu_s\nu_t}\;\;\ell^2(k_F r_{12})\right)$$

$$= \frac{1}{2}\,\rho \int d^3r\;\;\vartheta(r)\;\;\left(1 - \frac{\ell^2(k_F r_{12})}{\nu_s\nu_t}\right)$$

The expectation value of $V = \sum_i' v_{ij}$ can be calculated using the two-body distribution function:

$$\boxed{\frac{\langle\Psi|\sum_{i<j}' \vartheta(r_{ij})|\Psi\rangle}{N} = \frac{1}{2}\rho\int d^3r\,\vartheta(r)\,g(r)}$$

where

$$g(r) = \frac{N(N-1)}{\rho^2}\frac{\int d\Omega_{12}\,\Psi^*(\bar{r}_1,\ldots,\bar{r}_N)\,\Psi(\bar{r}_1,\ldots,\bar{r}_N)}{\int d\Omega\,\Psi^*\Psi}$$

for the free Fermi sea:

$$\boxed{g(r) = 1 - \frac{\ell^2(k_F r)}{\nu}}$$

$\nu = 4$ nuclear matter

$\ell^2(0) = 1$

If $\nu = 1$ (polarized neutron matter)

$g(0) = 0$, they can not be in the same place. Do not see the contact interaction

$\nu = 2$ $g(0) = \frac{1}{2}$ $\Rightarrow$ n° of total spin states 4

n° of forbidden states 2

$\nu = 4$ $g(0) = \frac{3}{4}$ $\Rightarrow$ n° of total states 16

n° of forbidden states 4

$\rightarrow$ number of allowed states $= 12$

$g(0) = \frac{12}{16} = \frac{3}{4}$ ok

Integrating first over coordinates

$$\frac{\langle V\rangle}{N} = \frac{1}{N}\,\frac{1}{2}\sum_{\alpha,\beta} \langle \varphi_\alpha(1)\varphi_\beta(2)\,|\,v(r)\,|\,\varphi_\alpha(1)\varphi_\beta(2) - \varphi_\beta(1)\varphi_\alpha(2)\rangle$$

$$\langle \varphi_\alpha(1)\varphi_\beta(2)\,|\,v(r)\,|\,\varphi_\alpha(1)\varphi_\beta(2) - \varphi_\beta(1)\varphi_\alpha(2)\rangle =$$

$$= \langle \vec{k}_1 m_{s_1} m_{t_1}, \vec{k}_2 m_{s_2} m_{t_2}\,|\,v(r)\,|\,\vec{k}_1 m_{s_1} m_{t_1}, \vec{k}_2 m_{s_2} m_{t_2} - \vec{k}_2 m_{s_2} m_{t_2}\,\vec{k}_1 m_{s_1} m_{t_1}\rangle$$

$$= \frac{1}{\Omega}\left[\int d^3r\, v(r) - \delta_{m_{s_1} m_{s_2}}\,\delta_{m_{t_1} m_{t_2}} \int d^3r\, v(r)\, e^{-i\bar{q}\bar{r}}\right]$$

$$\bar{q} = 2\bar{k}_r = \bar{k}_1 - \bar{k}_2$$

Now, one should do the summation over d^3k_1 and d^3k_2. It is convenient to perform a change of variables, to $\bar{k}_{cm}$ and $\bar{k}_r$ normalized such that:

$$\int_{k_1\le k_F} d^3k_1 \int_{k_2\le k_F} d^3k_2 = \frac{4}{3}\pi k_F^3 \int W(k_r)\, k_r^2\, dk_r\, d\Omega_{k_r}$$

$$= \frac{4}{3}\pi k_F^3 \cdot 4\pi \int_{k_r\le k_F} W(k_r)\, k_r^2\, dk_r \qquad \vec{k}_r = \frac{\bar{k}_1 - \bar{k}_2}{2}$$

with $\displaystyle W(k_r) = 8\left(1 - \frac{3}{2}\left(\frac{k_r}{k_F}\right) + \frac{1}{2}\left(\frac{k_r}{k_F}\right)^3\right)$

Checking the normalization:

$$4\pi\int W(k_r)\, k_r^2\, dk_r = \frac{4}{3}\pi k_F^3$$

$$\frac{\langle V \rangle}{N} = \frac{1}{2}\frac{1}{N}\frac{\Omega^2}{(2\pi)^6}\int d^3k_1 \int d^3k_2 \sum_{\substack{ms_1 ms_2 \\ mt_1 mt_2}}'$$

$$\frac{1}{\Omega}\left[\int d^3r\, \vartheta(r) - \delta_{ms_1 ms_2}\delta_{mt_1 mt_2}\int d^3r\, \vartheta(r)\, e^{-i2\bar{k}_r \bar{r}}\right]$$

$$= \frac{1}{2}\frac{1}{N}\frac{\Omega}{(2\pi)^3}\frac{4}{3}k_F^3\pi\frac{1}{(2\pi)^3}\frac{4}{3}\pi k_F^3\, \nu^2 \int d^3r\, \vartheta(r)$$

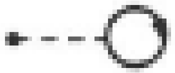
$$- \frac{1}{2}\frac{1}{N}\frac{\Omega}{(2\pi)^3}\overbrace{\frac{1}{(2\pi)^3}\nu\frac{4}{3}\pi k_F^3}^{\rho}\int_{k_r \le k_F} W(k_r)\, k_r^2\, dk_r\, d\Omega_{k_r}$$

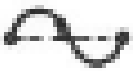
$$\cdot \int d^3r\, \vartheta(r)\, e^{-i2\bar{k}_r\bar{r}}$$

$$= \frac{1}{2}\frac{1}{\cancel{\rho}}\cancel{\rho}\,\rho\int d^3r\, \vartheta(r) - \frac{1}{2}\frac{1}{\cancel{\rho}}\cancel{\rho}\frac{1}{(2\pi)^3}4\pi\int dk_r\, W(k_r)\, \vartheta(k_r)$$

$$\boxed{\begin{aligned} \frac{\langle V\rangle}{N} &= \frac{1}{2}\rho\int d^3r\, \vartheta(r) - \frac{1}{4\pi^2}\int dk_r\, W(k_r)\, \vartheta(k_r) \\ &\text{where}\quad \vartheta(k_r) = \int d^3r\, \vartheta(r)\, e^{-i2k_r} \\ W(k_r) &= 8\left(1 - \frac{3}{2}\left(\frac{k_r}{k_F}\right) + \frac{1}{2}\left(\frac{k_r}{k_F}\right)^3\right) \end{aligned}}$$

Interaction of one particle with all the others





$$U(\vec{k}_1) = \frac{1}{\nu} \sum_{m_{s_1} m_{t_1}} U(k_1, m_{s_1}, m_{t_1}) = \frac{1}{\nu} \frac{V}{(2\pi)^3} \int_{k_2 < k_F} d^3k_2 \sum_{\substack{m_{s_1} m_{t_1} \\ m_{s_2} m_{t_2}}}$$

average

$$\langle \vec{k}_1 m_{s_1} m_{t_1}, \vec{k}_2 m_{s_2} m_{t_2} | \vartheta(r) | \vec{k}_1 m_{s_1} m_{t_1}, \vec{k}_2 m_{s_2} m_{t_2} - \vec{k}_2 m_{s_2} m_{t_2}, \vec{k}_1 \cdots \rangle$$

$$\langle \vec{k}_1 m_{s_1} m_{t_1}, \vec{k}_2 m_{s_2} m_{t_2} | \vartheta(r) | \vec{k}_1 m_{s_1} m_{t_1}, \vec{k}_2 m_{s_2} m_{t_2} - \vec{k}_2 m_{s_2} m_{t_2}, \vec{k}_1 m_{s_1} m_{t_1} \rangle$$

$$= \frac{1}{V} \left[\int d^3r\, \vartheta(r) - \delta_{m_{s_1} m_{s_2}} \delta_{m_{t_1} m_{t_2}} \int d^3r\, \vartheta(r)\, e^{-i\vec{q}\,\vec{r}} \right]$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{q} = \vec{k}_1 - \vec{k}_2 = 2\vec{k}_r$$

one can perform the summations over spin and isospin

$$\sum_{\substack{m_{s_1} m_{t_1} \\ m_{s_2} m_{t_2}}} 1 = \mathrm{Tr}(I) = \nu^2 \qquad \sum_{\substack{m_{s_1} m_{t_1} \\ m_{s_2} m_{t_2}}} \delta_{m_{s_1} m_{s_2}} \delta_{m_{t_1} m_{t_2}} = \nu = \mathrm{Tr}(P_\sigma P_c)$$

Therefore,

$$U(\vec{k}_1) = \frac{1}{\nu} \frac{V}{(2\pi)^3} \int_{k_2 \leq k_F} d^3k_2 \, \frac{1}{V} \left(\nu^2 \int d^3r \, \vartheta(r) - \nu \int d^3r \, \vartheta(r) \, e^{-i \vec{q}\,\vec{r}} \right)$$

$$\vec{q} = \vec{k}_1 - \vec{k}_2$$

The direct term is easy

$$\frac{1}{\nu} \frac{\cancel{V}}{(2\pi)^3} \int d^3k_2 \, \frac{1}{\cancel{V}} \nu^2 \int d^3r \, \vartheta(r) = \frac{\nu}{(2\pi)^3} \frac{4}{3} \pi k_F^3 \int d^3r \, \vartheta(r) = \rho \int d^3r \, \vartheta(r)$$

the exchange contribution:

$$-\frac{1}{(2\pi)^3}\int_{k_2\le k_F} d^3k_2 \int d^3r\, e^{-i\bar{k}_1\bar{r}}\, e^{i\bar{k}_2\bar{r}}\, \vartheta(r)$$

we can perform the integral over k_2

$$\int_{k_2\le k_F} d^3k_2\, e^{i\bar{k}_2\bar{r}} = \frac{4\pi}{r}\left[\frac{\sin k_F r}{r^2} - \frac{k_F \cos k_F r}{r}\right]$$

$$= 4\pi k_F^3 \frac{1}{(k_F r)}\left[\frac{\sin k_F r}{(k_F r)^2} - \frac{\cos k_F r}{(k_F r)}\right]$$

$$= 4\pi k_F^3 \frac{j_1(k_F r)}{k_F r} = (2\pi)^3 \frac{\rho}{\nu} \frac{3 j_1(k_F r)}{k_F r}$$

therefore:

$$-\frac{1}{\cancel{(2\pi)^3}}\int d^3r\, \cancel{(2\pi)^3}\, \frac{\rho}{\nu} \frac{3 j_1(k_F r)}{k_F r}\, e^{-i\bar{k}_1\vec{r}}\, \vartheta(r) =$$

$$= -\frac{\rho}{\nu} 4\pi\cdot 3\int dr\, \cancel{r^2}\, \frac{j_1(k_F r)}{k_F \cancel{r}} \frac{\sin k_1 r}{k_1 \cancel{r}}\, \vartheta(r)$$

$$= -\frac{\rho}{\nu}\frac{12\pi}{k_F k_1}\int_0^\infty dr\, j_1(k_F r)\sin(k_1 r)\, \vartheta(r)$$

Finally, the single particle potential is reduced to a one-dimensional integral.

$$\boxed{U(k_1) = 4\pi\rho\int dr\, r^2\, \vartheta(r) - \frac{\rho}{\nu}\frac{12\pi}{k_F k_1}\int dr\, j_1(k_F r)\sin(k_1 r)\, \vartheta(r)}$$

Normalized to volume

$$\langle \bar r_1 \bar r_2 | \bar k_1 \bar k_2 \; S \; T \; M_S \; M_T \rangle = \frac{1}{\sqrt{V}} e^{i \bar k_1 \bar r_1} \frac{1}{\sqrt{V}} e^{i \bar k_2 \bar r_2} \chi^S_{M_S} \lambda^T_{M_T}$$

not antisymmetrized.

$$\langle \bar R, \bar r | k_1 k_2 \; S \; T \; M_S \; M_T \rangle \Rightarrow \langle \bar R \; \bar r | \bar k_{CM} \; \bar k_r \; \chi^S_{M_S} \; \lambda^T_{M_T} \rangle$$

$$\vec R = \frac{1}{2} (\vec r_1 + \vec r_2) \qquad \vec k_{CM} = \vec k_1 + \vec k_2$$

$$\vec r = \vec r_2 - \vec r_1 \qquad \vec k_r = (k_2 - k_1)\frac{1}{2}$$

Jacobian

$$\frac{\partial(\bar R, \bar r)}{\partial(\bar r_1, \bar r_2)} = 1$$

Convenient to use explicitly the center of mass and the relative momentum

The antisymmetrization operator:

$$\hat A = \frac{1}{\sqrt{2}} (1 - \hat P_{12}) \qquad \hat P_{12} = P_r P_\sigma P_\tau$$

$$P_\sigma = \frac{1 + \bar\sigma_1 \bar\sigma_2}{2} \qquad P_\sigma \chi^S_{M_S} = (-1)^{S+1} \chi^S_{M_S}$$

$$P_\tau = \frac{1 + \bar\tau_1 \bar\tau_2}{2} \qquad P_\tau \lambda^T_{M_T} = (-1)^{1+T} \lambda^T_{M_T}$$

$$\langle \vec R \, | \vec k_{CM} \rangle = \frac{1}{\sqrt{V}} e^{i \bar k_{CM} \bar R}$$

$$P_r \, | \vec k_{CM} \rangle = | \bar k_{CM} \rangle$$

P_r does not affect the center of mass

$$P_r \, | \vec k_r \rangle = | -\vec k_r \rangle$$

if normalized to volume, $\langle \vec r \, | \bar k_r \rangle = \frac{1}{\sqrt{\Omega}} e^{i \bar k_r \bar r}$

$$\langle \bar R \; \bar r \, | \bar k_1 \bar k_2 \; S \; T \; M_S \; M_T \rangle = \langle \bar R \; \bar r \, | \bar k_{CM} \; \bar k_r \; S \; T \; M_S \; M_T \rangle$$

$$\langle \vec R\, \vec r \,|\, \vec k_{CM}\, \vec k_r\, S\, T\, M_S\, M_T \rangle =$$

$$= \frac{1}{V^{1/2}}\, e^{i \vec k_{CM} \vec R}\, \frac{1}{V^{1/2}}\, \underbrace{4\pi \sum_{lm} i^l\, j_l(kr)\, Y^*_{lm}(\hat k)\, Y_{lm}(\hat r)}_{e^{i \vec k_r \vec r}}\ \chi^S_{M_S}\, \lambda^T_{M_T}$$

Now we antisymmetrize and normalize.

$$|\vec k_1\, \vec k_2\, S\, T\, M_S\, M_T\rangle_a = \hat A\, |\vec k_1\, \vec k_2\, S\, T\, M_S\, M_T\rangle$$

$$= \hat A\, |\vec k_{CM}\, \vec k_r\, S\, T\, M_S\, M_T\rangle =$$

$$\frac{1}{\sqrt2}\left[\, |\vec k_{CM}\, \vec k_r\, S\, T\, M_S\, M_T\rangle - (-1)^{S+T}\, |\vec k_{CM}\, -\vec k_r\, S\, T\, M_S\, M_T\rangle \right]$$

then

$$\langle \vec R\, \vec r \,|\, \vec k_{CM}\, \vec k_r\, S\, T\, M_S\, M_T \rangle =$$

$$= \frac{1}{V^{1/2}}\, e^{i \vec k_{CM} \vec R} \cdot \frac{1}{V^{1/2}}\, 4\pi \sum_{lm} i^l\, j_l(kr) \left(1-(-1)^{l+S+T}\right) \frac{1}{\sqrt2}\, Y^*_{lm}(\hat k)\, Y_{lm}(\hat r)\, \chi^S_{M_S}\, \lambda^T_{M_T}$$

$$U(k_1) = \frac{1}{\nu} \sum_{\substack{m_{s_1}\\ m_{t_1}}} U(k_1, m_{s_1}, m_{t_1}) =$$

$$= \frac{1}{\nu} \frac{\Omega}{(2\pi)^3} \int_{k_2 \le k_F} d^3k_2 \sum_{\substack{m_{s_1} m_{t_1}\\ m_{s_2} m_{t_2}}} \langle \vec{k}_1 m_{s_1} m_{t_1} \cdot \vec{k}_2 m_{s_2} m_{t_2} | V(r) | \vec{k}_1 m_{s_1} m_{t_1} \vec{k}_2 m_{s_2} m_{t_2} \rangle_a$$

$$= \frac{1}{\nu} \frac{\Omega}{(2\pi)^3} \int_{k_2 \le k_F} d^3k_2 \sum_{\substack{S M_S\\ T M_T}} \langle \bar{k}_1 \bar{k}_2\, S T M_S M_T | V(r) | \bar{k}_1 \bar{k}_2\, S T M_S M_T \rangle_a$$

$$= \frac{1}{\nu} \frac{\Omega}{(2\pi)^3} \int_{k_2 \le k_F} d^3k_2 \quad \cancel{\frac{1}{\Omega} \int e^{-i k_{CM} \bar{R}} e^{i k_{CM} \bar{R}} d^3R}$$

$$\frac{1}{\Omega} \int r^2 dr \sum_{\substack{S M_S\\ T M_T\\ \ell m_\ell\\ \ell' m'_\ell}} (4\pi)^2 \frac{1}{\sqrt{2}} \left(1-(-1)^{\ell+S+T}\right) \frac{1}{\sqrt{2}} \left(1-(-1)^{\ell'+S+T}\right)$$

$$i^{-\ell}\, i^{+\ell'}\, j_\ell(k_r r)\, V(r)\, j_{\ell'}(k_r r)$$

$$\underbrace{\int Y_{\ell m}(\hat{r})\, Y^*_{\ell' m'}(\hat{r})\, d\Omega_r}_{\delta_{\ell\ell'}\delta_{mm'}} \quad Y^*_{\ell m}(\hat{k}_r)\, Y_{\ell' m'}(\hat{k}_r)$$

$$\underbrace{\langle \chi^S_{M_S} | \chi^S_{M_S} \rangle}_{1} \quad \underbrace{\langle \lambda^T_{M_T} | \lambda^T_{M_T} \rangle}_{1}$$

$$= \frac{1}{\nu} \frac{\cancel{\Omega}}{(2\pi)^3} \int_{k_2 \le k_F} d^3k_2 \frac{1}{\cancel{\Omega}} (4\pi)^2 \frac{1}{2} \sum_{\ell T S} \left(1-(-1)^{\ell+T+S}\right) \int dr\, r^2 j_\ell(k_r r) V(r)\, j_\ell(k_r r)$$

$$\underbrace{\sum_{m_\ell} Y^*_{\ell m}(\hat{k}_r)\, Y_{\ell m}(\hat{k}_r)}_{\frac{2\ell+1}{4\pi}} \quad \underbrace{\sum_{m_S} 1}_{2S+1} \quad \underbrace{\sum_{m_T} 1}_{2T+1}$$

$$U(k_1) = \frac{1}{\nu} \frac{4\pi}{(2\pi)^3} \sum_{lTS} \left(1-(-1)^{l+T+S}\right)(2l+1)(2T+1)(2S+1)$$

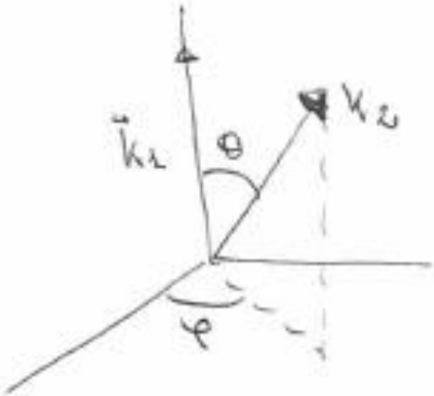
$$\int_{k_2 \le k_F} d^3k_2 \int dr\, r^2\, j_l(kr)\, V(r)\, j_l(kr)$$

Now, we use $$V^{l}_{k_r,k_r} = \frac{2}{\pi}\int r^2 dr\, j_l(k_r r)\, V(r)\, j_l(k_r r)$$

$$U(k_1) = \frac{1}{\nu} \frac{1}{4\pi} \sum_{SlT} \left(1-(-1)^{l+S+T}\right)(2l+1)(2S+1)(2T+1)$$

$$\int_{k_2 \le k_F} d^3k_2\; V^{l}_{k_r,k_r} \qquad \vec{k}_r = \frac{\vec{k}_2-\vec{k}_1}{2}$$

How to perform the integral over d^3k_2



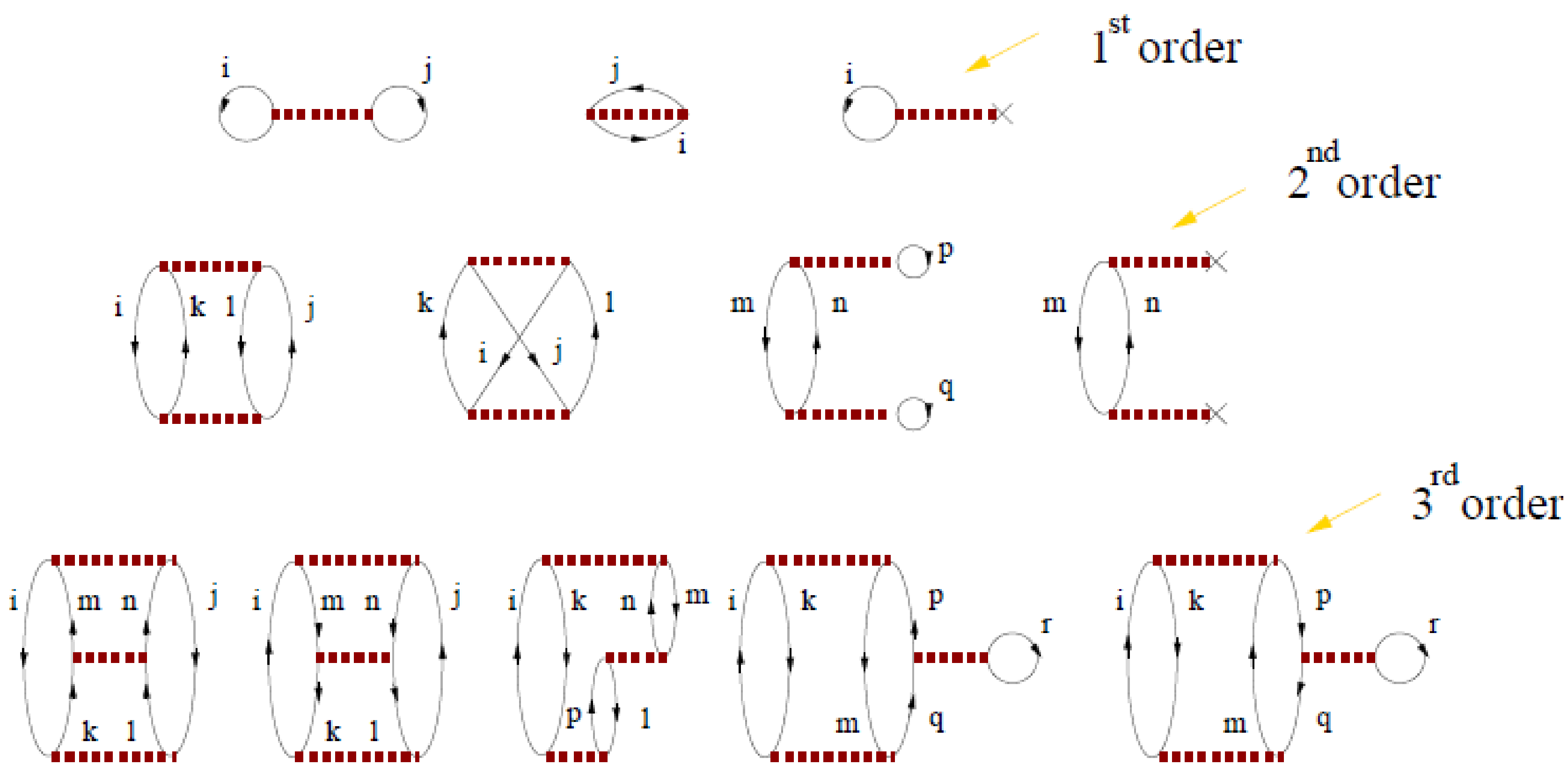
For a given $\vec{k}_2 \Rightarrow (k_2, \theta, \varphi)$

we have $k_r = \frac{1}{2}\sqrt{k_1^2 + k_2^2 - 2k_1k_2\cos\theta}$

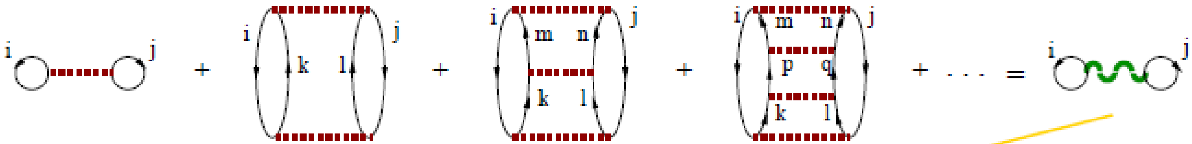
$$U(k_1) = \frac{1}{\nu}\,\frac{1}{4\pi}\,2\pi \sum_{S\ell T}\left(1-(-1)^{\ell+S+T}\right)(2\ell+1)(2S+1)(2T+1)\int_{k_2\le k_F} d^3k_2\; V^{\ell}_{k_r,k_r}$$

$$\boxed{U(k_1) = \frac{1}{\nu}\,\frac{1}{2}\sum_{S\ell T}\left(1-(-1)^{\ell+S+T}\right)(2\ell+1)(2S+1)(2T+1)\int_{k_2<k_F} dk_2\, k_2^2 \int_{-1}^{1} dx\; V^{\ell}\left(k_r = \frac{1}{2}\sqrt{k_1^2+k_2^2-2k_1k_2x}\right)}$$

Goldstone expansion



Ladder series



$$G = V + V\frac{Q}{\omega - H_0}V + V\frac{Q}{\omega - H_0}V\frac{Q}{\omega - H_0}V + \dots$$

$$\Rightarrow \quad G = V + V\frac{Q}{\omega - H_0}G$$ Bethe-Goldstone eq.

$$\boxed{T = V + V\frac{1}{\omega - K + i\eta}T}$$ Lippmann-Schwinger eq.

Lippman - Schwinger equation

$$\boxed{T = V + V \frac{1}{E - H_0 + i\varepsilon} T}$$

uncoupled case:

$$T_\ell(k, k'; E) = V_\ell(k, k') + \int_0^\infty dq\, q^2 \frac{V_\ell(k, q)\, T_\ell(q, k'; E)}{E - \frac{\hbar^2 q^2}{m} + i\varepsilon}$$

The reduce mass: $\mu = \frac{m}{2}$

and $$\boxed{V_\ell(k, k') = \frac{2}{\pi} \int dr\, r^2\, j_\ell(kr)\, V(r)\, j_\ell(k'r)}$$

How to make an integral (numerically)

$\int_a^\infty dk\, f(k)$ using Gauss points?

$$\int_a^\infty dk\, f(k) = \int_0^1 dx\, \frac{\pi}{2} \frac{1}{\cos^2 \frac{\pi}{2} x}\, f(a + tg\, \tfrac{\pi}{2} x)$$

The integral from $[a, \infty)$ is transformed in one integral in the interval $[0, 1)$, by means of the change of variables:

$$K = tg\, \frac{\pi}{2} x + a \qquad x \in [0, 1)$$

and $\dfrac{dk}{dx} = \dfrac{\pi}{2} \dfrac{1}{\cos^2 \frac{\pi}{2} x}$. Now we can't take No Gauss points in $[0, 1)$

$$\int_a^\infty dK\, f(k) = \sum_{i=1}^{N_G} w_i \frac{\pi}{2} \frac{1}{\cos^2 \frac{\pi}{2} x_i} f\left(a + tg \frac{\pi}{2} x_i\right)$$

Gauss weights in the interval $[0, 1]$ for N_G points.

I can also play with a constant if I don't go far enough!

$$K = b\, tg \frac{\pi}{2} x + a \qquad \frac{dK}{dx} = b \frac{\pi}{2} \frac{1}{\cos^2 \frac{\pi}{2} x}$$

$$\int_a^\infty dK\, f(k) = \sum_{i=1}^{N_G} w_i \frac{b\pi}{2} \frac{1}{\cos^2 \frac{\pi}{2} x} f\left(a + b\, tg \frac{\pi}{2} x_i\right)$$

$E<0$ no pole in the integral,

we can forget about $i\varepsilon$

Discretization of the integral: tangent map

$$V_\ell(k_i,k_m) = T_\ell(k_i,k_m) - \sum_j q_j^2\, w_j \frac{V_\ell(k_i,q_j)\,T_\ell(q_j,k_m)}{E - \frac{\hbar^2 q_j^2}{m}}$$

$k_i \quad i=1,---,N$

Matrix equation $N \times N$

$$\begin{bmatrix} 1-\dfrac{k_1^2\omega_1 V_e(k_1,k_1)}{E-\frac{\hbar^2k_1^2}{m}} & -\dfrac{k_2^2\omega_2 V_e(k_1,k_2)}{E-\frac{\hbar^2k_2^2}{m}} & \cdots & -\dfrac{k_N^2\omega_N V_e(k_1,k_N)}{E-\frac{\hbar^2k_N^2}{m}} \\ -\dfrac{k_1^2\omega_1 V_e(k_2,k_1)}{E-\frac{\hbar^2k_1^2}{m}} & 1-\dfrac{k_2^2\omega_2 V_e(k_2,k_2)}{E-\frac{\hbar^2k_2^2}{m}} & \cdots & \\ & & & \\ -\dfrac{k_1^2\omega_1 V_e(k_N,k_1)}{E-\frac{\hbar^2k_1^2}{m}} & & & 1-\dfrac{k_N^2\omega_N V_e(k_N,k_N)}{E-\frac{\hbar^2k_N^2}{m}} \end{bmatrix} \begin{bmatrix} T_e(k_1,k_m) \\ T_e(k_2,k_m) \\ \\ T_e(k_N,k_m) \end{bmatrix} = \begin{bmatrix} V_e(k_1,k_m) \\ V_e(k_2,k_m) \\ \vdots \\ V_e(k_N,k_m) \end{bmatrix}$$

What happens for positive energies?

For $E>0$, there is a pole in integral $\Rightarrow$

T becomes complex !

$$\frac{1}{E-H_0+i\eta} = P\left(\frac{1}{E-H_0}\right) - i\pi\,\delta(E-H_0)$$

Let's imagine that we have two integral equations that differ only in the propagator:

$$T = V + V\,P^{T}_{prop}\,T \qquad R = V + V\,P^{R}_{prop}\,R$$

then one can write an integral equation between T and R:

$$T = R + R\left\{P^{T}_{prop} - P^{R}_{prop}\right\}T$$

then one can write an integral equation between T and R:

$$T = R + R\left\{ P^{T}_{prop} - P^{R}_{prop} \right\} T$$

in our case

$$P^{T}_{prop} = \frac{1}{E - H_0 + i\eta} = P\frac{1}{E-H_0} - i\pi\,\delta(E - H_0)$$

$$P^{R}_{prop} = P\frac{1}{E-H_0}$$

$$P^{T}_{prop} - P^{R}_{prop} = -i\pi\,\delta(E-H_0)$$

$$R_\ell(E,k,k') = V_\ell(k,k') + P\int dq\, q^2 \frac{V_\ell(k,q)\, R_\ell(q,k')}{E - \frac{\hbar^2 q^2}{m}}$$

$$T_\ell(E,k,k') = R_\ell(E,k,k') - i\pi \int dq\, q^2\, R_\ell(E,k,q)\, \delta\left(E - \frac{\hbar^2 q^2}{m}\right) T_\ell(E,q,k')$$

we can manipulate the $\delta\left(E - \frac{\hbar^2 q^2}{m}\right)$

$$\delta(f(x)) = \left|\frac{df}{dx}\right|^{-1}_{x=x_0} \delta(x-x_0) \qquad \text{where } f(x_0)=0$$

in our case $\delta(E - E(q)) = \frac{m}{2\hbar^2 k_p}\, \delta(q - k_p)$

with $E = \frac{\hbar^2 k_p^2}{m}$ $\Rightarrow$

$$T_\ell(E,k,k') = R_\ell(E,k,k') - i\pi \int dq\, q^2\, R_\ell(E,k,q)\, \delta(q-k_p)\, \frac{m}{2\hbar^2 k_p}\, T_\ell(E,q,k')$$

finally

$$T_\ell(E,k,k') = R_\ell(E,k,k') - \frac{i\pi k_p m}{2\hbar^2} R_\ell(E,k,k_p)\, T_\ell(E,k_p,k')$$

If I take $k = k_p$ and $k' = k_p$ one gets on-shell matrix elements

$$T_\ell(E,k_p,k_p) = R_\ell(E,k_p,k_p) - \frac{i\pi k_p m}{2\hbar^2} R_\ell(E,k_p,k_p)\, T(E,k_p,k_p)$$

$$T_\ell(E,k_p,k_p) = \frac{R_\ell(E,k_p,k_p)}{1 + i\pi \frac{k_p m}{2\hbar^2} R_\ell(E,k_p,k_p)}$$

How to treat the principle value integral?

How to calculate $R_\ell(E, k_i, k_j)$?

$$R_\ell(E,k,k') = V_\ell(k,k') + P\int_0^\infty dq\, q^2\, V_\ell(k,q)\, \frac{1}{\frac{\hbar^2}{m}(k_p^2 - q^2)}\, R_\ell(q,k')$$

$E = \frac{\hbar^2}{m} k_p^2$ $\quad$ k_p is a pole

Now, we subtract a term, which principal value is zero and we get a smooth integrand

We subtract:

$$k_p^2 \lim_{q \to k_p} \left\{ \frac{q^2 - k_p^2}{D(q)} \right\} P \int_0^\infty \frac{dq}{q^2 - k_p^2}$$

where $P \int_0^\infty \frac{dq}{q^2 - k_p^2} = 0$ and $D(q) = \frac{\hbar^2 k_p^2}{m} - \frac{\hbar^2 q^2}{m}$

in this case (T matrix), the limit in the pole is simple:

L'Hopital

$$\lim_{q \to k_p} \left\{ \frac{q^2 - k_p^2}{D(q)} \right\} = \frac{2 k_p}{\left.\frac{d D(q)}{dq}\right|_{q=k_p}} = \frac{2 k_p}{-\frac{2 k_p \hbar^2}{m}} = -\frac{m}{\hbar^2}$$

In this way one gets an smooth integral and the mesh can ignore the Principle value

$$R_\ell(k,k') = V_\ell(k,k') + \mathcal{P}\int_0^\infty dq \, \frac{q^2 V_\ell(k,q) R_\ell(q,k') - k_p^2 V_\ell(k,k_p) R_\ell(k_p,k')}{\frac{\hbar^2}{m}(k_p^2 - q^2)}$$

Now, we discretize the equation, for $\{q_i, i=1 \ldots N\}$ with $q_i \neq k_p$, and write a system of $N+1$ linear equations for the points $\{q_i, k_p\}$

$$R_\ell(q_i, k_p) = V_\ell(q_i, k_p) + \sum_{j=1}^{N} \frac{q_j^2 w_j V_\ell(q_i, q_j) R_\ell(q_j, k_p)}{\frac{\hbar^2}{m}(k_p^2 - q_j^2)} - k_p^2 V_\ell(q_i, k_p) R_\ell(k_p, k_p) \sum_{j=1}^{N} \frac{w_j}{\frac{\hbar^2}{m}(k_p^2 - q_j^2)}$$

$i = 1, \dots, N$ equations

$$R_\ell(k_p, k_p) = V_\ell(k_p, k_p) + \sum_{j=1}^{N} \frac{q_j^2 w_j V_\ell(k_p, q_j) R_\ell(q_j, k_p)}{\frac{\hbar^2}{m}(k_p^2 - q_j^2)} - k_p^2 V_\ell(k_p, k_p) R_\ell(k_p, k_p) \sum_{j=1}^{N} \frac{w_j}{\frac{\hbar^2}{m}(k_p^2 - q_j^2)}$$

System with $N+1$ unknows $R_\ell(q_i, k_p)$, $i = 1, \dots, N$ and $R_\ell(k_p, k_p)$

In a matrix language

$$\left[\begin{array}{c|c} \delta_{ij} - \dfrac{V_\ell(q_i,q_j)\, q_j\, \omega_j^2}{\frac{\hbar^2}{m}(k_p^2-q_j^2)} & V_\ell(q_i,k_p)\,k_p^2 \displaystyle\sum_{j=1}^{N} \frac{\omega_j}{\frac{\hbar^2}{m}(k_p^2-q_j^2)} \\ \hline -\dfrac{V_\ell(k_p,q_j)\,q_j^2\,\omega_j}{\frac{\hbar^2}{m}(k_p^2-q_j^2)} & 1+V_\ell(k_p,k_p)\,k_p^2 \displaystyle\sum_{j=1}^{N} \frac{\omega_j}{\frac{\hbar^2}{m}(k_p^2-q_j^2)} \end{array}\right] \begin{bmatrix} R_\ell(q_1,k_p) \\ \vdots \\ R_\ell(q_N,k_p) \\ R_\ell(k_p,k_p) \end{bmatrix} = \begin{bmatrix} V_\ell(q_1,k_p) \\ \vdots \\ V_\ell(q_N,k_p) \\ V_\ell(k_p,k_p) \end{bmatrix}$$

Inverting this matrix one can determine $R_\ell(q_i,k_p)$ and $R_\ell(k_p,k_p)$ that will be on-shell.

One can also go directly to the T matrix by including the delta in the Integral and inverting a complex matrix

$$T_\ell(k,k_p) = V_\ell(k,k_p) + \int dq\, q^2 \frac{V_\ell(k,q)\, T_\ell(q,k_p)}{\frac{\hbar^2}{m}k_p^2 - \frac{\hbar^2 q^2}{m} + i\eta}$$

$$= V_\ell(k,k_p) + P\int dq\, q^2 \frac{V_\ell(k,q)\, T_\ell(q,k_p)}{\frac{\hbar^2}{m}k_p^2 - \frac{\hbar^2}{m}q^2} - i\pi \int dq\, q^2\, \delta\left(\frac{\hbar^2}{m}k_p^2 - \frac{\hbar^2}{m}q^2\right) V_\ell(k,q)\, T_\ell(q,k_p)$$

$$- \int dq\; k_p^2\, V_\ell(k,k_p)\, T_\ell(k_p,k_p) \frac{1}{\frac{\hbar^2}{m}(k_p^2 - q^2)}$$

$$= V_\ell(k,k_p) + P\int dq\, \frac{q^2 V_\ell(k,q)\, T_\ell(q,k_p) - k_p^2 V_\ell(k,k_p)\, T_\ell(k_p,k_p)}{\frac{\hbar^2}{m}\left(k_p^2 - q^2\right)}$$

$$- i\pi\, \frac{V_\ell(k,k_p)\, T_\ell(k_p,k_p)}{2\,\frac{\hbar^2}{m}\,k_p}$$

$$\left[\begin{array}{c|c}
\delta_{ij} - \dfrac{V_\ell(q_i,q_j)\, q_j^2\, w_j}{\frac{\hbar^2}{m}\left(k_p^2 - q^2\right)} & \begin{array}{l} + k_p^2\, V_\ell(q_i,k_p) \displaystyle\sum_j \dfrac{w_j}{\frac{\hbar^2}{m}\left(k_p^2 - q_j^2\right)} \\ + i\pi\, \dfrac{V_\ell(q_i,k_p)}{2\,\frac{\hbar^2}{m}\,k_p} \end{array} \\
\hline
- \dfrac{V_\ell(k_p,q_j)\, q_j^2\, w_j}{\frac{\hbar^2}{m}\left(k_p^2 - q_j^2\right)} & \begin{array}{l} 1 + V_\ell(k_p,k_p)\, k_p^2 \displaystyle\sum_{j=1}^{N} \dfrac{w_j}{\frac{\hbar^2}{m}\left(k_p^2 - q_j^2\right)} \\ + i\pi\, \dfrac{V_\ell(k_p,k_p)}{2\,\frac{\hbar^2}{m}\,k_p} \end{array}
\end{array}\right]
\begin{bmatrix} T_\ell(q_1,k_p) \\ \vdots \\ T_\ell(q_N,k_p) \\ T_\ell(k_p,k_p) \end{bmatrix}
=
\begin{bmatrix} V_\ell(q_1,k_p) \\ \vdots \\ V_\ell(q_N,k_p) \\ V_\ell(k_p,k_p) \end{bmatrix}$$

Some physics and some checks

$$T_\ell(E, k_p, k_p) = \frac{R_\ell(E; k_p, k_p)}{1 + i\pi \frac{k_p m}{2\hbar^2} R_\ell(E; k_p, k_p)}$$

on shell
$E = \frac{\hbar^2}{m} k_p^2$

$$\frac{1}{T_\ell(E; k_p, k_p)} \underset{E = \frac{\hbar^2}{m} k_p^2}{=} \frac{1 + i\pi \frac{k_p m}{2\hbar^2} R_\ell(E; k_p, k_p)}{R_\ell(E; k_p, k_p)}$$

$$\mathrm{Im}\left(\frac{2\hbar^2}{\pi m} \frac{1}{T_\ell(E; k_p, k_p)} \right) = k_p$$

Besidy :

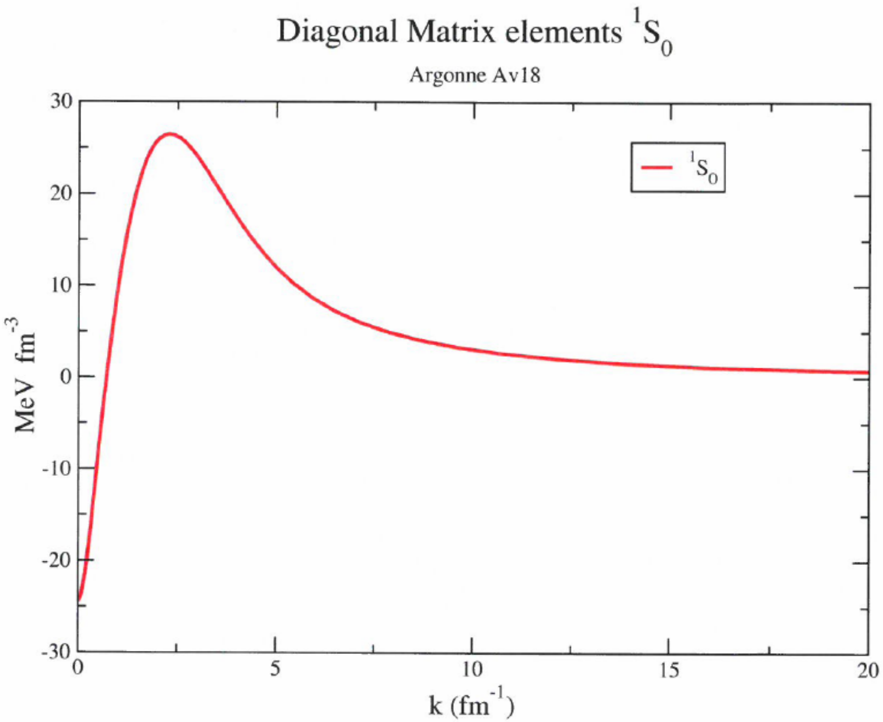
$$\underbrace{T_\ell(E; k_p, k_p)}_{MeV \cdot fm^3} \frac{\pi}{2} \frac{m}{\hbar^2} = \frac{1}{-k_p \cot \delta_\ell + i k_p}$$

$$\frac{1}{\frac{\pi}{2} \frac{m}{\hbar^2} T_\ell(E; k_p, k_p)} = -k_p \cot \delta_\ell(k_p) + i k_p$$

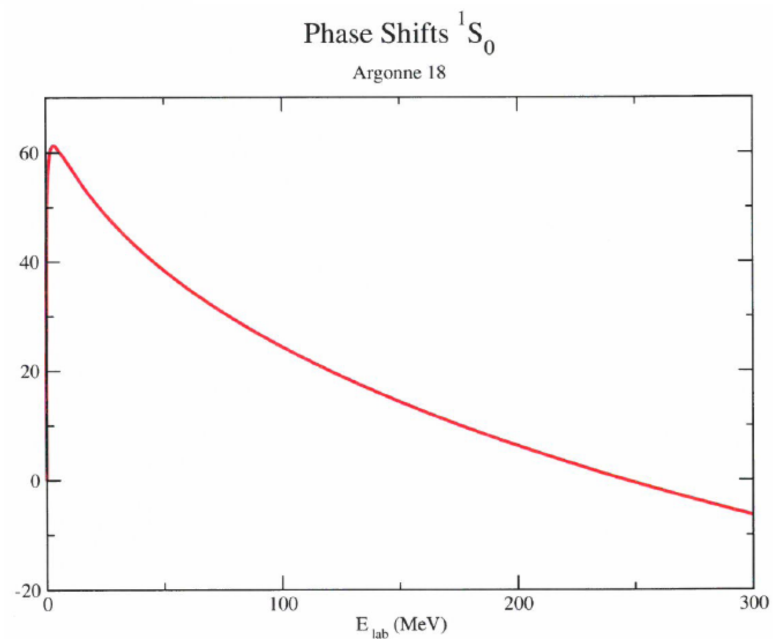
$$\mathrm{Re} \frac{1}{\frac{\pi}{2} \frac{m}{\hbar^2} T_\ell(E; k_p, k_p)} = -k_p \cot \delta_\ell(k_p)$$

$\ell = 0$ $\quad {}^1S_0$ $\quad \Rightarrow \quad \mathrm{Re} \dfrac{1}{\frac{\pi}{2} \frac{m}{\hbar^2} T_\ell(E; k, k)} \simeq \dfrac{1}{a_0} - \dfrac{1}{2} r_0 k^2$

$L = 0, S = 0, J = 0$

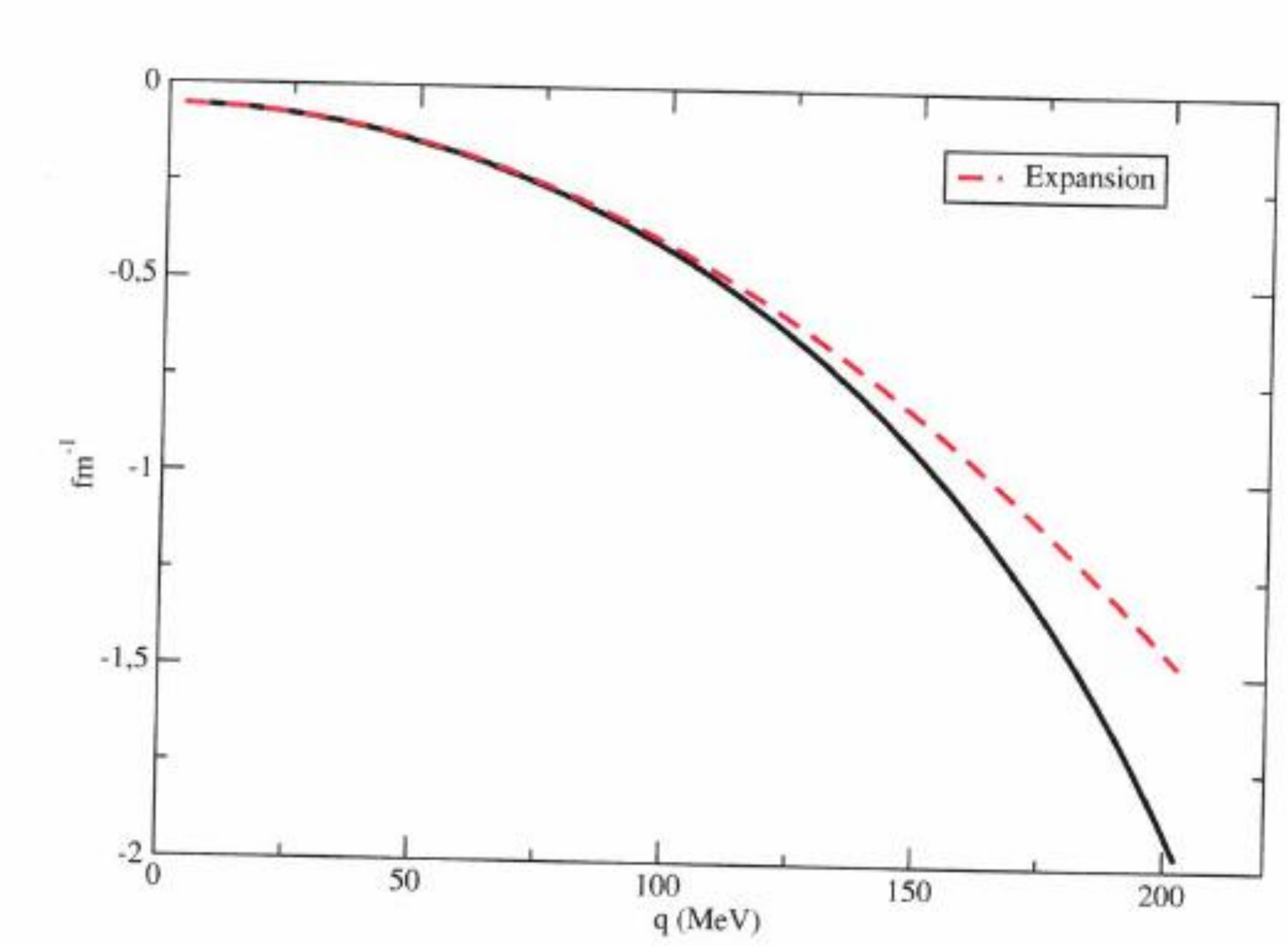


Diagonal Matrix elements $^1S_0$
Argonne Av18
$^1S_0$
MeV fm^{-3}
k (fm^{-1})
30
20
10
0
-10
-20
-30
0
5
10
15
20

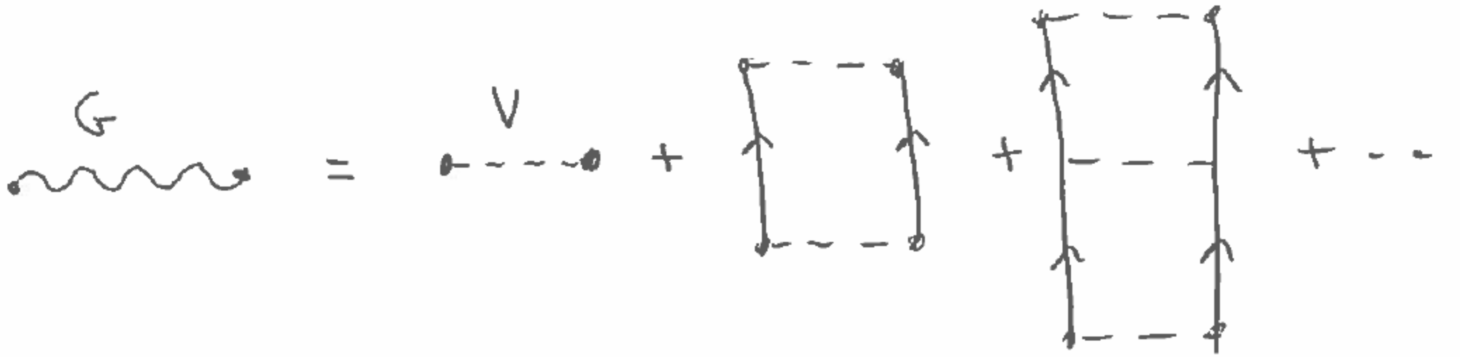


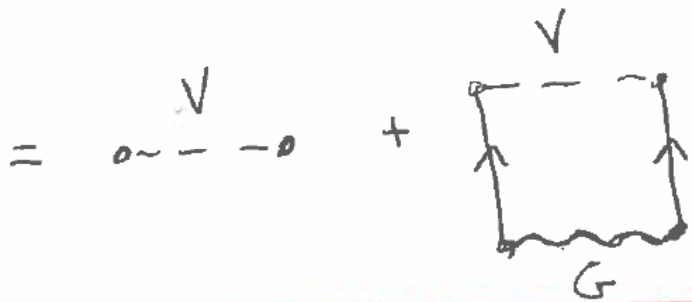
Phase Shifts $^1S_0$
Argonne 18
60
40
20
0
-20
0
100
200
300
E_{lab} (MeV)

Comparison with the expansión in terms of the scattering length and the Effective range.



Let's try to sum the ladder diagrams in the medium





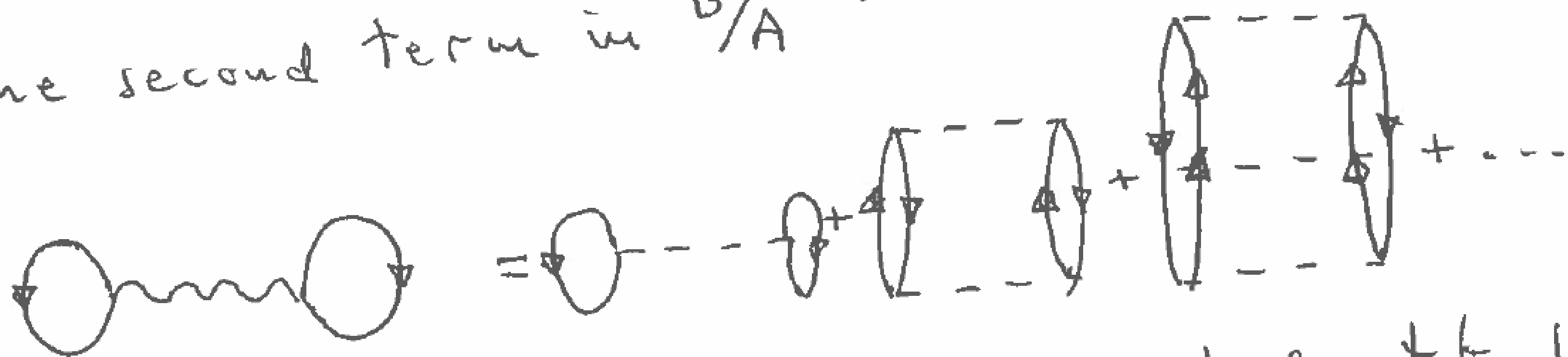
Ladder Goldstone diagrams contributing to the G-matrix

$$\boxed{G(E) = V + V\frac{Q}{E - H_0 + i\eta} G(E)}$$

$$\boxed{\frac{B}{A} = \sum_{\vec{k}} \frac{\hbar^2 k^2}{2m} + \frac{1}{2} \sum_{k,j<k_F} \langle k j \,|\, G(\varepsilon(k)+\varepsilon(j)) \,|\, k j \rangle_a}$$

$|kj\rangle$ antisymmetrized two-body states.

The second term in B/A.



an upgoing arrow denotes a particle state $k>k_F$

a downgoing arrow refers to a hole state $k<k_F$

First Term of the hole line expansión !

The single particle potential

The single particle energy in the intermediate states

$$\varepsilon(k) = \frac{\hbar^2 k^2}{2m} + \underbrace{\sum_{j<k_F} \langle kj | G(\varepsilon(k)+\varepsilon(j)) | kj \rangle}_{\text{single-particle potential}}$$

Single-particle potential U(k), which can be complex. One takes the real part.

If it is complex , for the propagation we use the real part!

To solve the G-matrix, it is convenient to express the two-body states in terms of the center of mass and relative momenta:

$$\langle \vec{R}\ \vec{r} \mid \vec{K}_{CM}\ \vec{k}_r \rangle = \frac{1}{\sqrt{2}}\ \frac{1}{V}\ e^{i\vec{K}_{CM}\vec{R}}\left(e^{i\vec{k}_r\vec{r}} - e^{-i\vec{k}_r\vec{r}}\right)$$

$$\langle \vec{k}_1 \vec{k}_2 \mid V(r) \mid \vec{k}_3 \vec{k}_4 \rangle = \frac{1}{V}\ \delta_{\vec{K}_{CM},\vec{K}'_{CM}}\ \langle \vec{k}_r \mid V \mid \vec{k}'_r \rangle$$

where the matrix elements on the relative momentum contain the antisymmetrization.

$$\langle \vec{k}_r \mid V \mid \vec{k}'_r \rangle = \frac{1}{2}\int d^3r \left[e^{i\vec{k}_r\vec{r}} - e^{-i\vec{k}_r\vec{r}}\right]^* V(r) \left[e^{i\vec{k}'_r\vec{r}} - e^{-i\vec{k}_r\vec{r}}\right]$$

Try to reduce the dimensionality of the integral! Use partial wave expansión!

$$\langle \vec{k}_r\, S M_S\, T M_T | G(K_{CM}, \Omega) | \vec{k}_r'\, S M_S\, T M_T \rangle$$

$$= \langle \vec{k}_r\, S M_S\, T M_T | V | \vec{k}_r'\, S M_S\, T M_T \rangle$$

$$+ \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \langle \vec{k}_r | V | \vec{q} \rangle \frac{\tilde{Q}(K, q)}{\Omega - \varepsilon\left(\left|\frac{\vec{K}_{CM}}{2} + \vec{q}\right|\right) - \varepsilon\left(\left|\frac{\vec{K}_{CM}}{2} - \vec{q}\right|\right) + i\eta}$$

$$\langle \vec{q} | G(K_{CM}, \Omega) | \vec{k}_r' \rangle$$

partial wave decomposition, introduction of the partial waves in the antisymmetric states:

$$\langle \vec{r} \,|\, \vec{k}_r\, S\, M_S\, T\, M_T \rangle = \frac{1}{\sqrt{2}} \left(e^{i \vec{k}_r \vec{r}} - (-1)^{S+T} e^{-i \vec{k}_r \vec{r}} \right) \chi^S_{M_S}\, \lambda^T_{M_T}$$

$\Big\downarrow$ angular momentum

$$\langle \vec{r} \,|\, \vec{k}_r\, S\, M_S\, T\, M_T \rangle = 4\pi \sum_{l m_l} j_l(k_r r)\, i^l\, Y_{l m_l}(\hat{r})\, Y^*_{l m_l}(\hat{k}_r)\, \frac{\left(1-(-1)^{L+S+T}\right)}{\sqrt{2}}\, \chi^S_{M_S}\, \lambda^T_{M_T}$$

partial wave antisymmetric state

$$\langle \vec{r} \,|\, K\, L\, M_L\, S\, M_S\, T\, M_T \rangle \equiv i^l \left(\frac{2}{\pi}\right)^{1/2} \frac{\left(1-(-1)^{L+S+T}\right)}{\sqrt{2}}\, K\, j_L(Kr)\, Y_{L M_L}(\hat{r})\, \chi^S_{M_S}\, \lambda^T_{M_T}$$

$$\langle k\, L\, M_L\, S\, M_S\, T\, M_T \,|\, k'\, L'\, M_L'\, S\, M_S\, T\, M_T\rangle =$$
$$\delta_{LL'}\, \delta_{M_L M_L'}\, \delta(k-k')\, \left(1-(-1)^{L+S+T}\right)$$

and the completeness relation:

$$\frac{1}{2} \sum_{L M_L} \int dk\, |k\, L\, M_L\, S\, M_S\, T\, M_T\rangle\langle k\, L\, M_L\, S\, M_S\, T\, M_T| = 1$$

Remember! We want to obtain the partial wave decomposition of

$$\langle \vec{k}_r\, S\, M_S\, T\, M_T \,|\, V(r) \,|\, \vec{k}_r'\, S\, M_S\, T\, M_T\rangle$$

to this end we will use the completeness relation shown above. We still need the following overlaps between antisymmetric states

$$\langle \vec{k}_r\, S\, M_S\, T\, M_T \,|\, k\, L\, M_L\, S\, M_S\, T\, M_T\rangle =$$
$$= \left(1-(-1)^{L+S+T}\right) (2\pi)^{3/2}\, \frac{\delta(k-k_r)}{k_r}\, Y_{L M_L}(\hat{k}_r)$$

and for a central local potential:

$$\boxed{\begin{aligned} &\langle k_r\, L\, M_L\, S\, M_S\, T\, M_T \,|\, V(r) \,|\, k_r'\, L'\, M_L'\, S\, M_S'\, T\, M_T\rangle = \\ &= \delta_{LL'}\, \delta_{M_L M_L'}\, \delta_{M_S M_S'} \left(1-(-1)^{L+S+T}\right) k_r\, k_r' \\ &\qquad \left(\frac{2}{\pi}\right) \int r^2 dr\, j_L(k_r r)\, V(r)\, j_L(k_r' r) \end{aligned}}$$

Definition:

$$\boxed{V_L(k_r, k_r') = \frac{2}{\pi}\int r^2 dr\, j_l(k_r r)\, V(r)\, j_l(k_r' r)}$$

Remember:

$$\int dr\, r^2 j_l(kr)\, j_l(k'r) = \frac{\pi}{2k^2}\delta(k-k')$$

Finally, the partial wave decomposition of the antisymmetric matrix element reads

$$\boxed{\begin{aligned} &\langle \vec{k}_r\, S\, M_S\, T\, M_T \,|\, V(r) \,|\, \vec{k}_r'\, S\, M_S'\, T\, M_T \rangle = \\ &= \delta_{M_S M_S'} (2\pi)^3 \sum_{L M_L} \left(1-(-1)^{L+S+T}\right) Y_{L M_L}(\hat{k}_r) \\ &\qquad\qquad Y^*_{L M_L}(\hat{k}_r')\, V_L(k_r, k_r') \end{aligned}}$$

Now, introducing the partial wave decomposition in the G-matrix equation, (one gets rid of the factor $\frac{1}{2}$ and the $\frac{1}{(2\pi)^3}$) and reduces the integral to one dimension.

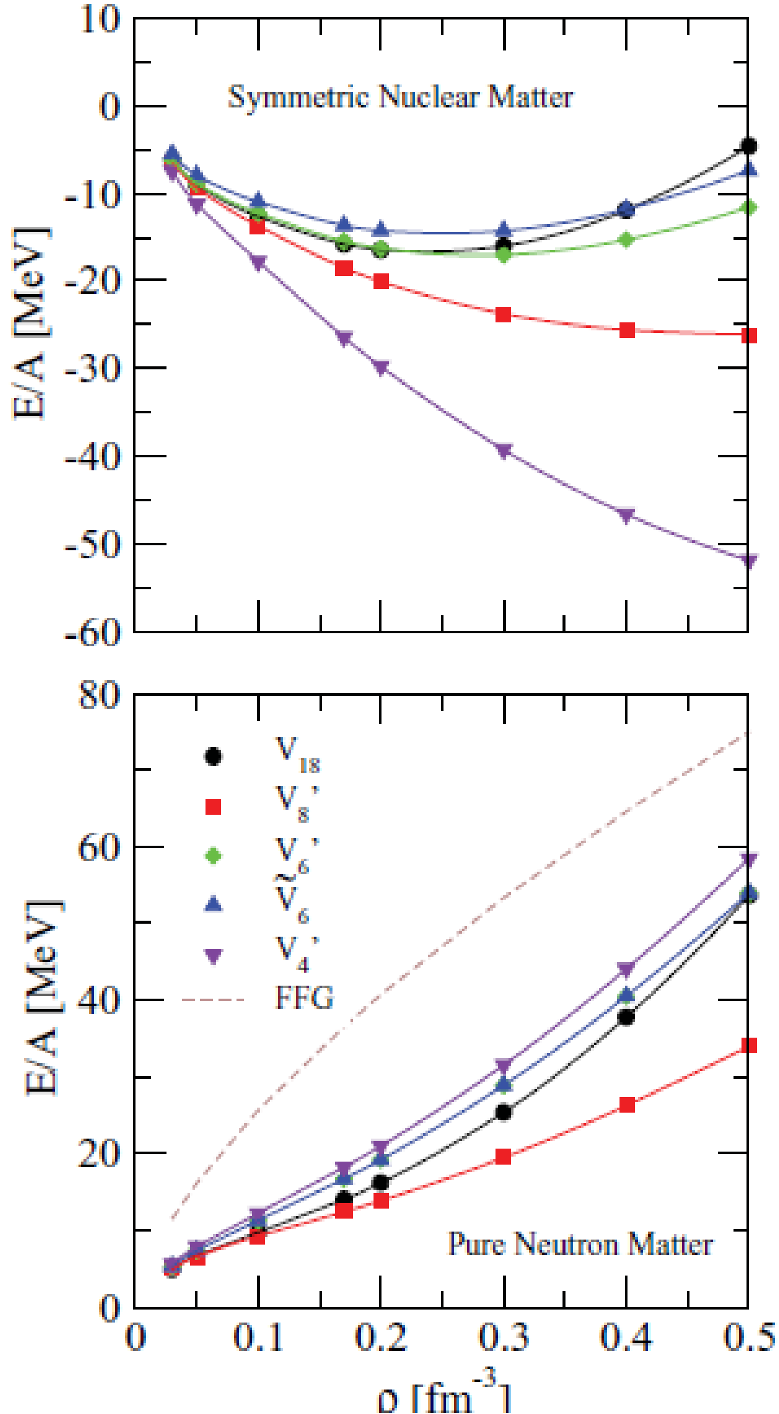
$$G_L(k_r, k_r', K_{CM}, \Omega) = V_L(k_r, k_r') + \int_0^\infty q^2 dq \; V_L(k_r, q) \frac{\tilde{Q}(K_{CM}, q)}{\Omega - \varepsilon_+(K_{CM}, q) - \varepsilon_-(K_{CM}, q) + i\eta} G_L(q, k_r', K_{CM}, \Omega)$$

Similar equation to the T-matrix

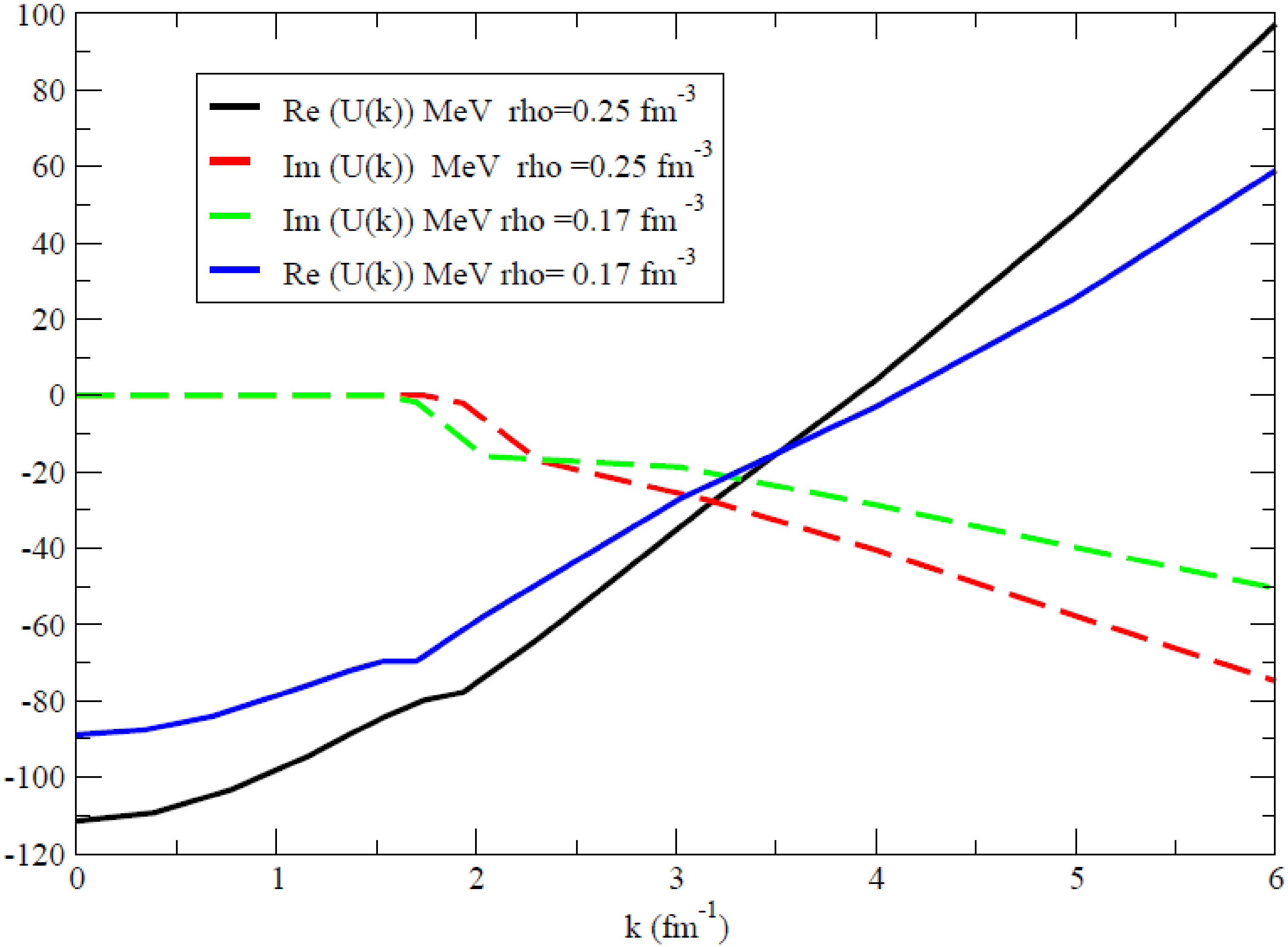
$$\varepsilon_\pm(K_{CM}, q) = \varepsilon\left(\left|\frac{\vec{K}_{CM}}{2} \pm \vec{q}\right|\right)$$

$$\left|\frac{\vec{k}_{CM}}{2} \pm \vec{q}\right| = \frac{1}{4} k_{CM}^2 + q^2 \pm \frac{1}{\sqrt{3}} k_{CM}\, q \; \bar{\varphi}^{3/2}(k_{CM}, q)$$

$G_L(k_r, k_r', k_{CM}, \Omega)$ has a singularity if the energy parameter Ω equals the available energy of the two-particle state in the denominator $\Longrightarrow$ For those energies $G_L(k_r, k_r', k_{CM}, \Omega)$ becomes complex


Symmetric Nuclear Matter
E/A [MeV]
10
0
-10
-20
-30
-40
-50
-60
80
60
40
20
0
V_{18}
V_8'
V_6'
$\tilde{V}_6$
V_4'
FFG
Pure Neutron Matter
0
0.1
0.2
0.3
0.4
0.5
ρ [fm^{-3}]

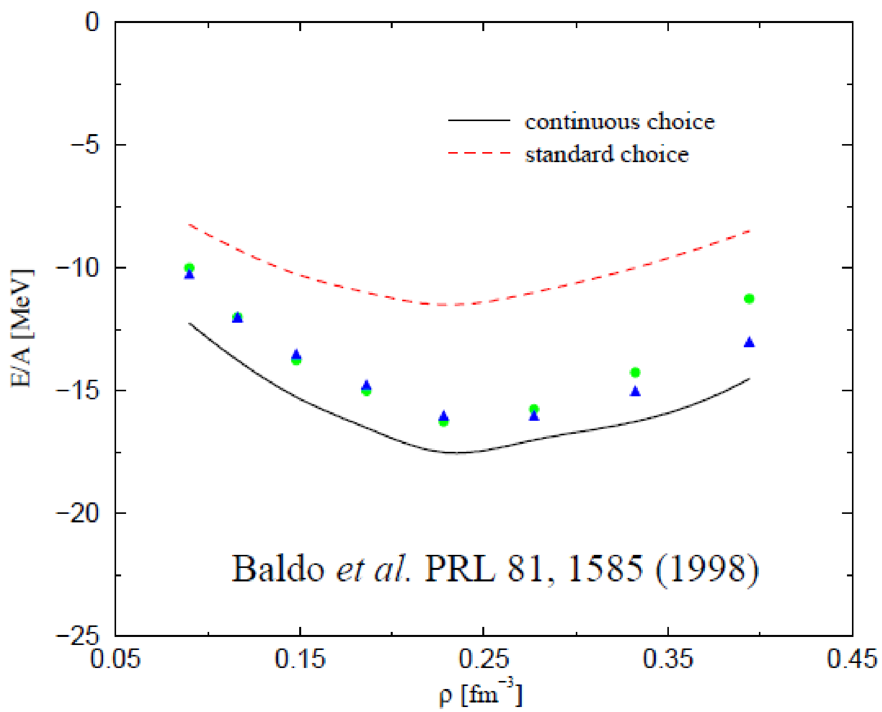
Single Particle potential

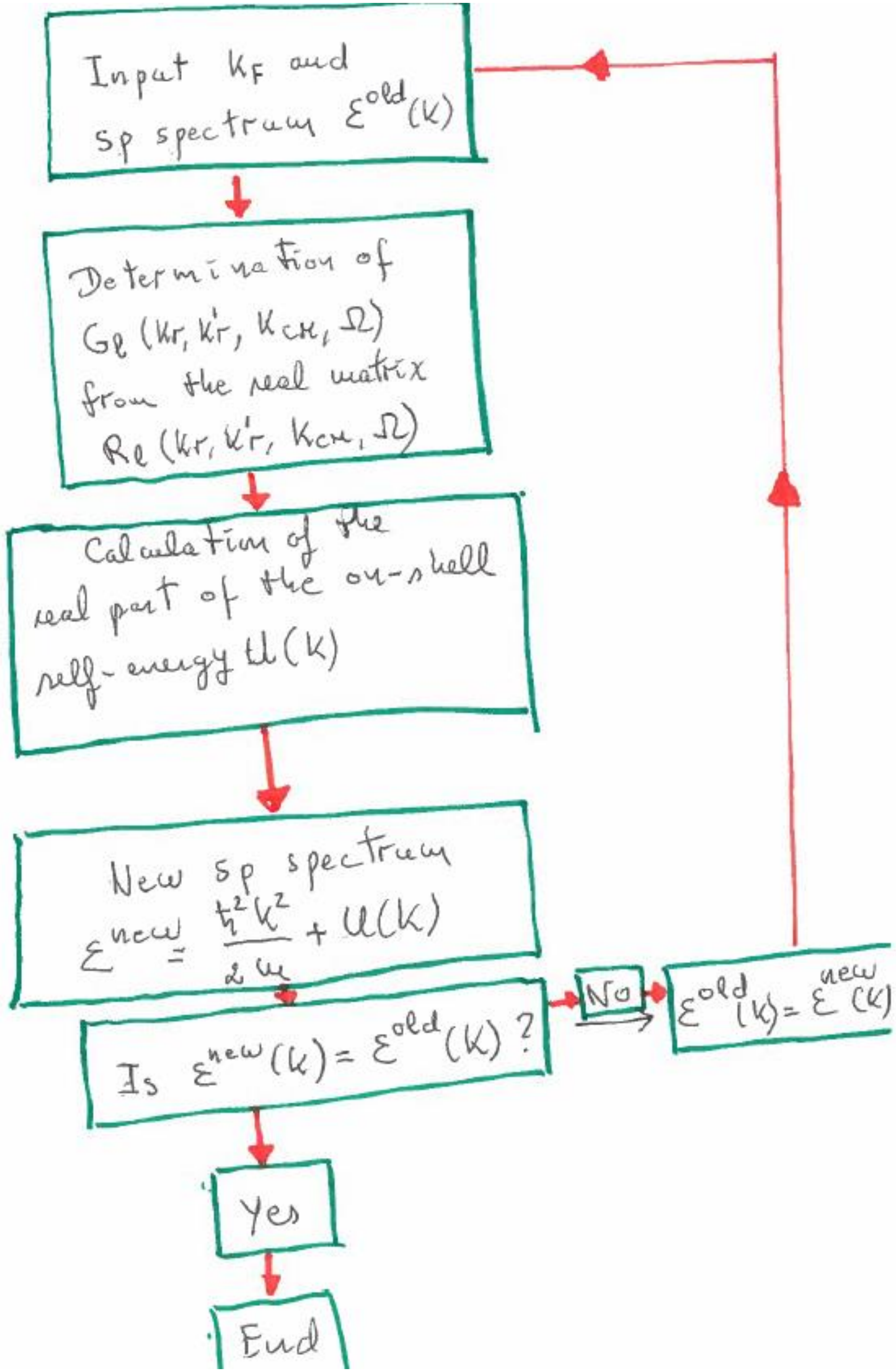


$$U_i^{BHF} = \mathrm{Re} \sum_{j<A} \langle \alpha_i \alpha_j | G(\omega) | \alpha_i \alpha_j \rangle_{\mathcal{A}}$$

$$\text{Standard Choice: } U_i = \begin{cases} U_i^{BHF} & \text{for } k < k_{F_{B_i}} \\ 0 & \text{for } k > k_{F_{B_i}} \end{cases}$$

$$\text{Continuous Choice: } U_i = U_i^{BHF} \ \forall \ k$$




Input k_F and sp spectrum $\varepsilon^{old}(k)$
Determination of $G_\ell(k_r, k'_r, K_{CM}, \Omega)$ from the real matrix $R_\ell(k_r, k'_r, K_{CM}, \Omega)$
Calculation of the real part of the on-shell self-energy $U(k)$
New sp spectrum $\varepsilon^{new} = \frac{\hbar^2 k^2}{2m} + U(k)$
Is $\varepsilon^{new}(k) = \varepsilon^{old}(k)$?
No
$\varepsilon^{old}(k) = \varepsilon^{new}(k)$
Yes
End

Average Pauli Operator

Pauli operator $\quad Q(\vec{k}_1, \vec{k}_2) = \Theta(|\vec{k}_1| - k_F)\,\Theta(|\vec{k}_2| - k_F)$

$\vec{k}_1$ and $\vec{k}_2$ in the LAB system.

In terms of the center of mass momentum

$\vec{K}_{CM} = \vec{k}_1 + \vec{k}_2$ and the relative momentum $\vec{k}_r = \frac{\vec{k}_1 - \vec{k}_2}{2}$

$$\vec{k}_1 = \frac{1}{2}\vec{K}_{CM} + \vec{k}_r$$

$$\vec{k}_2 = \frac{1}{2}\vec{K}_{CM} - \vec{k}_r$$

$$Q(\vec{K}_{CM}, \vec{k}_r) = \Theta\left(\left|\frac{1}{2}\vec{K}_{CM} + \vec{k}_r\right| - k_F\right)\Theta\left(\left|\frac{1}{2}\vec{K}_{CM} - \vec{k}_r\right| - k_F\right)$$

useful to define $\vec{P} = \frac{1}{2}\vec{K}_{CM}$, then $\vec{P}$ is taken along the z axis, and we take the angular

average

average

$$\bar{Q}(P, k_r) = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int \sin\theta \, d\theta \; Q(\vec{P}, \vec{k}_r)$$

$$Q(P, \vec{k}_r) = \Theta(|\vec{P} + \vec{k}_r| - k_F)\,\Theta(|\vec{P} - \vec{k}_r| - k_F)$$

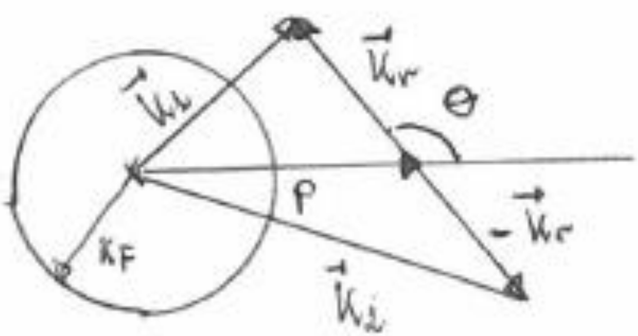
in terms of P, and make average we take $\vec{P}$ along the z axis, over $d\hat{k}_r$

We must distinguish two regions

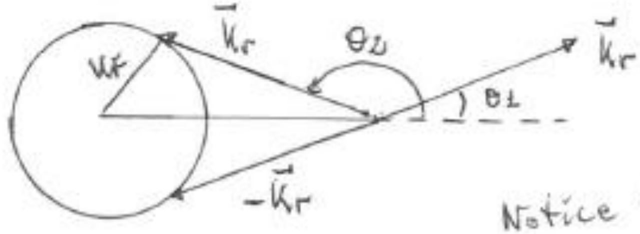
a) $P > k_F$

* If $k_r < P - k_F \Rightarrow$ We are always outside the Fermi sphere
$\Rightarrow \bar{Q} = 1$

* If $k_r > P + k_F \Rightarrow$ outside the Fermi sphere $\Rightarrow \bar{Q} = 1$



$P > k_F$ and $P - k_F < k_r < P + k_F$



$Q = 1$ for $\hat{k}_r$ such that $\theta_1 \le \theta < \theta_2$

Notice that for $\theta < \theta_1$, $\vec{k}_2$ is inside the Fermi sphere, and that for $\theta > \theta_2$, $\vec{k}_1 = \vec{P} + \vec{k}_r$ is inside the Fermi sphere.

$$\tilde{Q} = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta = \frac{2\pi}{4\pi} \Big[\cos\theta\Big]_{\theta_2}^{\theta_1}$$

At θ_2 $\quad \vec{P} + \vec{k}_r = \vec{k}_F \rightarrow k_F^2 = P^2 + k_r^2 + 2 k_r P \cos\theta_2$

$$\cos\theta_2 = \frac{k_F^2 - P^2 - k_r^2}{2 P k_r}$$

For θ_1, $\quad \vec{P} - \vec{k}_r = k_F \rightarrow k_F^2 = P^2 + k_r^2 - 2 P k_r \cos\theta_1$

$$\cos\theta_1 = \frac{P^2 + k_r^2 - k_F^2}{2 P k_r}$$

$$\Rightarrow \quad \tilde{Q} = \frac{1}{2}\left[\frac{P^2 + k_r^2 - k_F^2}{2 P k_r} - \frac{k_F^2 - P^2 - k_r^2}{2 P k_r}\right] = \frac{P^2 + k_r^2 - k_F^2}{2 P k_r}$$

Average Pauli operator

Average $\overline{Q}_{PP}(k_r, P)$, we include the subindices PP because the intermediate states are two particles above the Fermi level.

A) $P > k_F$

$0 \leq k_r \leq P - k_F$ $\quad \overline{Q}_{PP} = 1$

$P + k_F < k_r$ $\quad \overline{Q}_{PP} = 1$

$P - k_F \leq k_r \leq P + k_F$ $\quad \overline{Q}_{PP} = \dfrac{P^2 + k_r^2 - k_F^2}{2 P k_r}$

B) $P < k_F$

$0 \leq k_r \leq \sqrt{k_F^2 - P^2}$ $\quad \overline{Q}_{PP} = 0$

$k_F + P \leq k_r$ $\quad \overline{Q}_{PP} = 1$

$\sqrt{k_F^2 - P^2} \leq k_r \leq k_F + P$

$$\overline{Q}_{PP} = \frac{P^2 + k_r^2 - k_F^2}{2 P k_r}$$

Effective forces

$$H = \sum_i \frac{p_i^2}{2m} + \sum_{i<j} v(r_{ij})$$

A very simple one for nuclear matter.

Skyrme force. Many types.

$$\boxed{v(r_{ij}) = \left(t_0 + \frac{1}{6} t_3 \rho^\gamma\right) \delta(\bar{r}_i - \bar{r}_j)}$$

* Contact force
* only s-wave

$\gamma = 1/3$

$t_0 = -1794.$ $MeV \cdot fm^3$

$t_3 = 12817$ $MeV \cdot fm^{3+\gamma}$

$\frac{\hbar^2}{m} = 41.4687$ $MeV \cdot fm^2$

Two-body matrix elements

$$\langle \vec{k}_1 m_{s_1} m_{t_1}, \bar{k}_2 m_{s_2} m_{t_2} | (t_0 + \tfrac{1}{6} t_3 \rho^\gamma)\, \delta(\bar{r}_1 - \bar{r}_2) |$$

$$\bar{k}_1 m_{s_1} m_{t_1}, \vec{k}_2 m_{s_2} m_{t_2} - \bar{k}_2 m_{s_2} m_{t_2}, \bar{k}_1 m_{s_1} m_{t_1} \rangle =$$

$$\langle \bar{k}_1 \bar{k}_2 | (t_0 + \tfrac{1}{6} t_3 \rho^\gamma)\, \delta(\bar{r}_1 - \bar{r}_2) | \bar{k}_1 \bar{k}_2 \rangle \underbrace{\langle m_{s_1} m_{s_2} | m_{s_1} m_{s_2} \rangle}_{1}$$

$$\underbrace{\langle m_{t_1} m_{t_2} | m_{t_1} m_{t_2} \rangle}_{1}$$

$$- \langle k_1 k_2 | (t_0 + \tfrac{1}{6} t_3 \rho^\gamma)\, \delta(\bar{r}_1 - \bar{r}_2) | \bar{k}_2 \bar{k}_1 \rangle$$

$$\underbrace{\langle m_{s_1} m_{s_2} | m_{s_2} m_{s_1} \rangle}_{\delta_{m_{s_1} m_{s_2}}} \underbrace{\langle m_{t_1} m_{t_2} | m_{t_2} m_{t_1} \rangle}_{\delta_{m_{t_1} m_{t_2}}}$$

Spatial part normalized to volume

Direct part

$$\langle \bar{k}_1 \bar{k}_2 | (t_0 + \frac{1}{6} t_3 \rho^\gamma)\, \delta(\bar{r}_1 - \bar{r}_2) \,|\bar{k}_1 \bar{k}_2\rangle =$$

$$(t_0 + \frac{1}{6} t_3 \rho^\gamma) \frac{1}{\Omega^2} \int d^3 r_1\, d^3 r_2\, e^{-i\bar{k}_1 \bar{r}_1} e^{-i\bar{k}_2 \bar{r}_2} \delta(\bar{r}_1 - \bar{r}_2)\, e^{i\bar{k}_1 \bar{r}_1} e^{i\bar{k}_2 \bar{r}_2}$$

$$= (t_0 + \frac{1}{6} t_3 \rho^\gamma) \frac{1}{\Omega^2} \int d^3 r_1\, d^3 r_2\, \delta(\bar{r}_1 - \bar{r}_2) = \frac{1}{\Omega} (t_0 + \frac{1}{6} t_3 \rho^\gamma)$$

Exchange part

$$\langle \bar{k}_1 \bar{k}_2 | (t_0 + \frac{1}{6} t_3 \rho^\gamma)\, \delta(\bar{r}_1 - \bar{r}_2) \,|\bar{k}_2 \bar{k}_1\rangle =$$

$$= (t_0 + \frac{1}{6} t_3 \rho^\gamma) \frac{1}{\Omega^2} \int d^3 r_1\, d^3 r_2\, e^{-ik_1 r_1} e^{-ik_2 r_2} \delta(\bar{r}_1 - \bar{r}_2)\, e^{i\bar{k}_2 \bar{r}_1} e^{i\bar{k}_1 \bar{r}_2} =$$

$$= (t_0 + \frac{1}{6} t_3 \rho^\gamma) \frac{1}{\Omega^2} \int d^3 r_1\, d^3 r_2\, e^{i(\bar{k}_2 - \bar{k}_1)(\bar{r}_1 - \bar{r}_2)} \delta(\bar{r}_1 - \bar{r}_2) =$$

$$= \frac{1}{\Omega} (t_0 + \frac{1}{6} t_3 \rho^\gamma)$$

$$\boxed{\begin{aligned}&\langle k_1\, m_{s_1}\, m_{t_1}, k_2\, m_{s_2}\, m_{t_2} \,|\, (t_0 + \tfrac{t_3}{6}\rho^\gamma)\,\delta(\vec r_1 - \vec r_2)\,| \\ &\qquad \vec k_1\, m_{s_1} m_{t_1}, \vec k_2\, m_{s_2} m_{t_2} - \vec k_2\, m_{s_2} m_{t_2}, \vec k_1\, m_{s_1} m_{t_1}\rangle \\ &= \frac{1}{\Omega}\left(t_0 + \frac{t_3}{6}\rho^\gamma\right)\left(1 - \delta_{m_{s_1} m_{s_2}}\, \delta_{m_{t_1} m_{t_2}}\right)\end{aligned}}$$

<u>Expectation value. HF</u> $\quad \dfrac{\langle \phi_{FS} | V | \phi_{FS} \rangle}{N}$?

$$\frac{\langle \phi_{FS} | V | \phi_{FS} \rangle}{N} = \frac{1}{2} \frac{1}{N} \sum_{\alpha\beta}' \langle \alpha\beta | \vartheta(12) | \alpha\beta - \beta\alpha \rangle$$

$$= \frac{1}{2} \frac{1}{N} \sum_{\substack{m_{s_1} m_{s_2} \\ m_{t_1} m_{t_2}}}' \frac{\Omega}{(2\pi)^3} \int_{k_1 \le k_F} d^3k_1 \; \frac{\Omega}{(2\pi)^3} \int_{k_2 \le k_F} d^3k_2$$

$$\frac{1}{\Omega} \left(t_0 + \frac{t_3}{6} \rho^\gamma \right) \left(1 - \delta_{m_{s_1} m_{s_2}} \delta_{m_{t_1} m_{t_2}} \right)$$

$$= \frac{1}{2} \frac{1}{N} \frac{\Omega}{(2\pi)^3} \frac{\Omega}{(2\pi)^3} \frac{4}{3} \pi k_F^3 \frac{4}{3} \pi k_F^3 \frac{1}{\Omega} \left(t_0 + \frac{t_3}{6} \rho^\gamma \right)$$

$$\left[\sum' 1 - \sum' \delta_{m_{s_1} m_{s_2}} \delta_{m_{t_1} m_{t_2}} \right]$$

$$\sum_{\substack{m_{s_1} m_{s_2} \\ m_{t_1} m_{t_2}}}' 1 = \mathrm{Tr}(I) = (\deg)^2$$

deg = 4 for nuclear matter

deg = 2 for neutron matter

$$\sum_{m_{s_1} m_{s_2} \, m_{t_1} m_{t_2}}' \delta_{m_{s_1} m_{s_2}} \delta_{m_{t_1} m_{t_2}} = \sum' \langle m_{s_1} m_{s_2} | P_\sigma | m_{s_1} m_{s_2} \rangle \langle m_{t_1} m_{t_2} | P_t | m_{t_1} m_{t_2} \rangle = \mathrm{Tr}(P_\sigma P_t) P_t$$

$$\mathrm{Tr}\left(P_\sigma P_t\right) = \deg \qquad P_\sigma = \frac{1+\vec{\sigma}_1\vec{\sigma}_2}{2} \quad P_t = \frac{1+\vec{t}_1\vec{t}_2}{2}$$

Now,

$$= \frac{1}{2}\,\frac{1}{N}\,\frac{\Omega}{(2\pi)^3}\,\frac{1}{(2\pi)^3}\,\frac{4}{3}\pi k_F^3\,\frac{4}{3}\pi k_F^3\left(t_0 + \frac{t_3}{6}\rho^\gamma\right)\deg^2\left[1-\frac{1}{\deg}\right]$$

Remember:

$$\rho = \frac{\deg}{(2\pi)^3}\int_{k\leq k_F} d^3k \Rightarrow \frac{\deg}{(2\pi)^3}\,\frac{4}{3}\pi k_F^3 = \rho \Rightarrow \boxed{\rho = \frac{\deg\, k_F^3}{6\pi^2}}$$

$$= \frac{1}{2}\,\frac{1}{\rho}\,\rho^2\left(t_0 + \frac{t_3}{6}\rho^\gamma\right)\frac{3}{4}$$

$$\boxed{\frac{\langle\phi_{FS}|V|\phi_{FS}\rangle}{N} = \frac{1}{2}\rho\left(t_0 + \frac{t_3}{6}\rho^\gamma\right)\frac{3}{4}}$$

Total energy

$$e = \frac{E}{N} = \frac{1}{N}\left[\langle \phi_{FS} | T | \phi_{FS} \rangle + \langle \phi_{FS} | V | \phi_{FS} \rangle\right]$$

$$= \frac{3}{5}\frac{\hbar^2 k_F^2}{2m} + \frac{1}{2}\rho\left(t_0 + \frac{t_3}{6}\rho^\gamma\right)\frac{3}{4}$$

$$\boxed{e(\rho) = \frac{\hbar^2}{2m}\frac{3}{5}\left(\frac{3\pi^2}{2}\right)^{2/3}\rho^{2/3} + \frac{1}{2}\rho\left(t_0 + \frac{t_3}{6}\rho^\alpha\right)\frac{3}{4}}$$

Derivatives

$$\boxed{P = -\left(\frac{\partial E}{\partial \Omega}\right)_N = \rho^2 \frac{\partial e(\rho)}{\partial \rho}}$$

$$\boxed{\mu = \left(\frac{\partial E}{\partial N}\right)_\Omega = e(\rho) + \frac{P(\rho)}{\rho}}$$

$$K_T = -\frac{1}{\Omega}\left(\frac{\partial \Omega}{\partial P}\right)_N \Rightarrow K_T^{-1} = -\left(\frac{\partial P}{\partial \Omega}\right)_N \Omega = \left(\frac{\partial P}{\partial \rho}\right)\rho$$

$$c_s^2 = \frac{K_T^{-1}}{\rho\, w} \Rightarrow \boxed{\frac{c_s}{c} = \sqrt{\frac{K_T^{-1}}{\rho\, w\, c^2}}}$$

$$K_T^{-1} = 2\rho^2 \frac{\partial e(\rho)}{\partial \rho} + \rho^3 \frac{\partial^2 e(\rho)}{\partial \rho^2}$$

$$\boxed{P(\rho) = \rho^2 \frac{de(\rho)}{d\rho} = \frac{\hbar^2}{2m} \frac{2}{5} \left(\frac{3\pi^2}{2}\right)^{2/3} \rho^{5/3} + \frac{3}{4} \frac{1}{2} \rho^2 t_0 + (\gamma+1) \rho^{\gamma+2} \frac{3}{8} \frac{t_3}{6}}$$

$$\boxed{\mu(\rho) = e(\rho) + \frac{P(\rho)}{\rho} = \frac{\hbar^2}{2m} \left(\frac{3\pi^2}{2}\right)^{2/3} \rho^{2/3} + \frac{3}{4} \rho t_0 + (\gamma+2) \rho^{\gamma+1} \frac{3}{8} \frac{t_3}{6}}$$

In symmetric nuclear matter, the single particle potential, i.e., the interaction of one nucleon of momentum $\vec{k}$ with all the other will not depend on the third component of isospin $\Rightarrow$ will be the same independently if it is a proton or a neutron, and will also not depend on the third component of spin.

Therefore I can do an average over spin and isospin: $\nu = \nu_s \nu_t$ $\begin{cases} 4 & \text{nuclear matter} \\ 2 & \text{neutron matter} \end{cases}$

$$U_{HF}(k) = \frac{1}{\nu} \sum_{\substack{\vec{k}_2, m_{s_1} m_{t_1} \\ m_{s_2} m_{t_2}}} \langle \vec{k}\, m_{s_1} m_{t_1}, \vec{k}_2\, m_{s_2} m_{t_2} | v(\vec{r}_{12}) | \underbrace{\vec{k}\, m_{s_1} m_{t_1}, \vec{k}_2 m_{s_2} m_{t_2}}_{\text{direct}} - \underbrace{\vec{k}_2 m_{s_2} m_{t_2}, \vec{k}\, m_{s_1} m_{t_1}}_{\text{exchange}} \rangle$$

for the simple Skyrme force that we are considering

$$= \frac{1}{\nu} \sum_{\substack{m_{s_1} m_{t_1} \\ m_{s_2} m_{t_2}}} \frac{\cancel{\Omega}}{(2\pi)^3} \int_{k_2 \le k_F} d^3k_2 \, \frac{1}{\cancel{\Omega}} \left(t_0 + \frac{t_3}{6} \rho^\gamma\right) \left(1 - \delta_{m_{s_1} m_{s_2}} \delta_{m_{t_1} m_{t_2}}\right)$$

$$U_{HF}(k) = \frac{1}{\cancel{\nu}}\left(t_0 + \frac{t_3}{6}\rho^\gamma\right) \underbrace{\frac{1}{(2\pi)^3}\,\frac{4}{3}\pi k_F^3\,\cancel{\nu}}_{\rho}\,\underbrace{\left(1-\frac{1}{\nu}\right)}_{3/4}$$

$$= \frac{3}{4}\rho\left(t_0 + \frac{t_3}{6}\rho^\gamma\right)$$

which is a constant independent of k.

Usually $U_{HF}(k)$ depends on k.

From $U_{HF}(k)$, one can recover $\frac{\langle\phi_{FS}|V|\phi_{FS}\rangle}{N}$

as all spin-isospin give the same contribution one can multiply by ν

$$\frac{\langle\phi_{FS}|V|\phi_{FS}\rangle}{N} = \frac{1}{2}\,\frac{\nu}{N}\sum_{\vec{k}}{}' U_{HF}(k)$$

$$= \frac{1}{2}\,\frac{\nu}{N}\,\frac{\Omega}{(2\pi)^3}\int d^3k \cdot \theta(k_F - k)\;\frac{3}{4}\rho\left(t_0 + \frac{t_3}{6}\rho^\gamma\right)$$

$$= \frac{1}{2}\,\frac{3}{4}\rho\left(t_0 + \frac{t_3}{6}\rho^\gamma\right)\frac{1}{\rho}\;\underbrace{\frac{\nu}{(2\pi)^3}\,\frac{4}{3}\pi k_F^3}_{\rho}$$

$$= \frac{1}{2}\,\frac{3}{4}\rho\left(t_0 + \frac{t_3}{6}\rho^\gamma\right)$$

$$\text{Is} \quad \varepsilon^{HF}(k) = \mu(\rho) \; ?$$

$$\varepsilon^{HF}(k) = \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{2}\right)^{2/3}\rho^{2/3} + \frac{3}{4}\rho t_0 + \frac{t_3}{6}\rho^{\gamma+1}\frac{3}{4}$$

$$\mu(\rho) = \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{2}\right)^{2/3}\rho^{2/3} + \frac{3}{4}\rho t_0 + \frac{t_3}{6}\rho^{\gamma+1}\frac{3}{4} + \gamma\rho^{\gamma+1}\frac{t_3}{6}\frac{3}{8}$$

Is the chemical potential equal to the Fermi energy?

The total energy:

$$E = \sum_i \frac{\hbar^2 k_i^2}{2m} n_i + \frac{1}{2} \sum_{i \neq j} \langle ij | v | ij - ji \rangle n_i n_j$$

$\Rightarrow$ the single-particle energy:

$$\varepsilon(i) = \frac{\delta E}{\delta n_i} = \frac{\hbar^2 k_i^2}{2m} + \sum_j \langle ij | v | ij - ji \rangle n_j$$

$$+ \frac{1}{2} \sum_{i \neq j} n_i n_j \langle ij | \frac{\delta v}{\delta n_i} | ij - ji \rangle$$

This is the rearrangement term.

v depends on the occupations through its dependence on the density

$\frac{\delta v}{\delta n_i}$? $\quad \rho = \frac{1}{\Omega} \sum n_i \Rightarrow \frac{\delta}{\delta n_i} = \frac{\delta \rho}{\delta n_i} \frac{\delta}{\delta \rho} = \frac{1}{\Omega} \frac{\delta}{\delta \rho}$

$$\frac{\delta v}{\delta n_i} = \frac{1}{\Omega} \frac{\delta}{\delta \rho} \left(\frac{1}{6} t_3 \rho^\gamma \delta(\bar{r}_{12}) \right) = \frac{1}{\Omega} \frac{1}{6} t_3 \gamma \rho^{\gamma - 1} \delta(\bar{r}_{12})$$

$$\langle ij | \frac{\delta v}{\delta n_i} | ij - ji \rangle =$$

$$\langle ij | \frac{1}{\Omega} \frac{1}{6} t_3 \gamma \rho^{\gamma-1} \delta(\bar{r}_{12}) | ij - ji \rangle =$$

$$= \frac{1}{\Omega} \frac{1}{\Omega^2} \int d^3 r_1 \, d^3 r_2 \, \frac{1}{6} t_3 \gamma \rho^{\gamma-1} \delta(\bar{r}_{12}) \left(1 - \delta_{m_{s_1} m_{s_2}} \delta_{m_{t_1} m_{t_2}}\right)$$

$$= \frac{1}{\Omega^2} \frac{1}{6} t_3 \gamma \rho^{\gamma-1} \left(1 - \delta_{m_{s_1} m_{s_2}} \delta_{m_{t_1} m_{t_2}}\right)$$

$$u^R(k) = \frac{1}{2} \sum_{i \neq j} n_i n_j \langle ij | \frac{\delta v}{\delta n_i} | ij - ji \rangle =$$

$$= \frac{1}{2} \frac{\cancel{\Omega^2}}{(2\pi)^6} \int d^3 k_1 \, d^3 k_2 \; \Theta(k_F - k_1) \, \Theta(k_F - k_2) \frac{1}{\cancel{\Omega^2}} \sum_{\substack{m_{s_1} m_{s_2} \\ m_{t_1} m_{t_2}}} \frac{1}{6} t_3 \gamma \rho^{\gamma-1} \left(1 - \delta_{m_{s_1} m_{s_2}} \delta_{m_{t_1} m_{t_2}}\right)$$

$$= \frac{1}{2} \rho^2 \frac{1}{6} t_3 \gamma \rho^{\gamma-1} \left(1 - \frac{1}{\nu}\right) = \frac{t_3}{6} \gamma \rho^{\gamma+1} \frac{3}{8}$$

$$\frac{\langle \phi_{FS} | V | \phi_{FS} \rangle}{N} = \frac{1}{2} \rho \int d^3 r \; \vartheta(r) \left(1 - \frac{\ell^2(k_F r)}{\nu}\right)$$

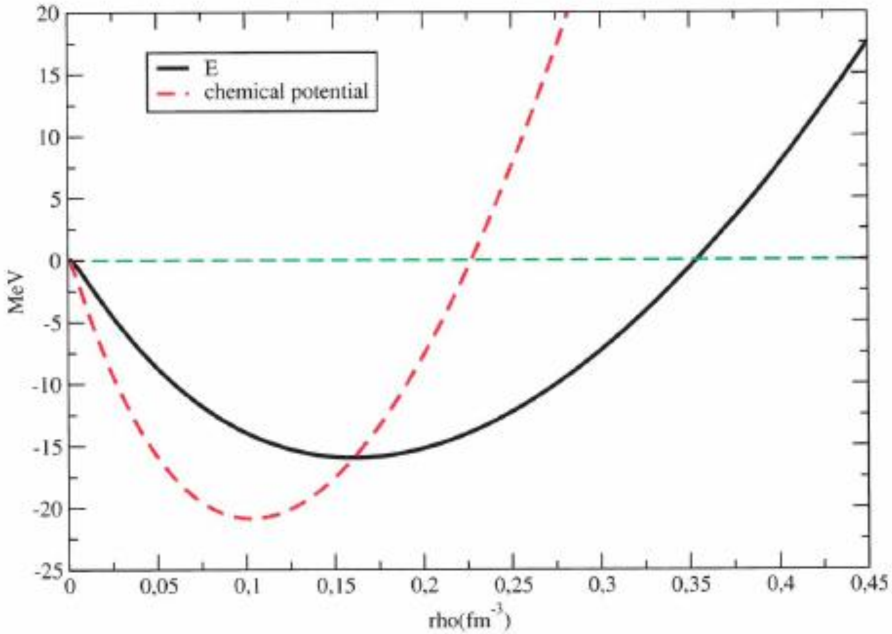
$$\vartheta(r) = \left(t_0 + \frac{1}{6} t_3 \rho^\gamma\right) \delta(\vec{r})$$

$$= \frac{1}{2} \rho \left(t_0 + \frac{1}{6} t_3 \rho^\gamma\right) \int d^3 r \; \delta(\vec{r}) \left(1 - \frac{\ell^2(k_F r)}{\nu}\right)$$

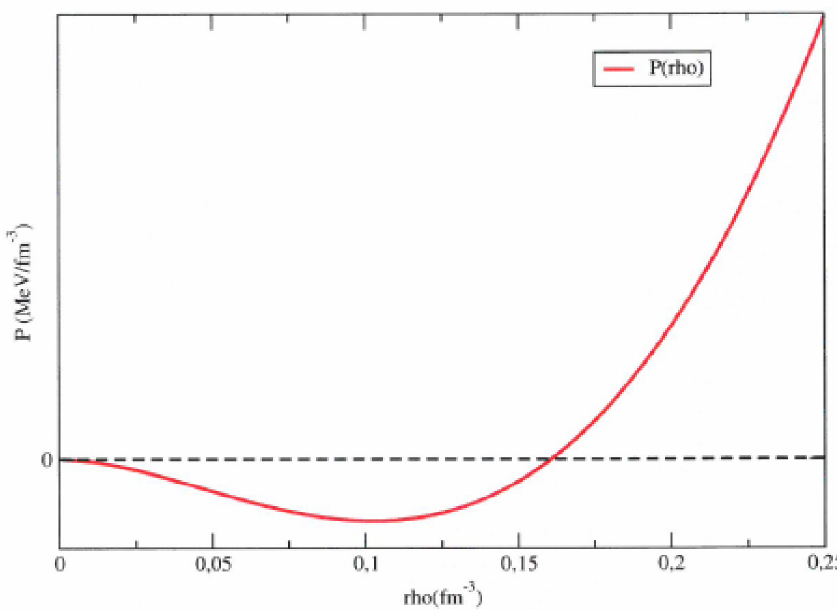
$$= \frac{1}{2} \rho \left(t_0 + \frac{1}{6} t_3 \rho^\gamma\right) \left(1 - \frac{\ell^2(0)}{\nu}\right)$$

$\ell^2(0) = 1$ $\quad \nu = 4$ nuclear matter

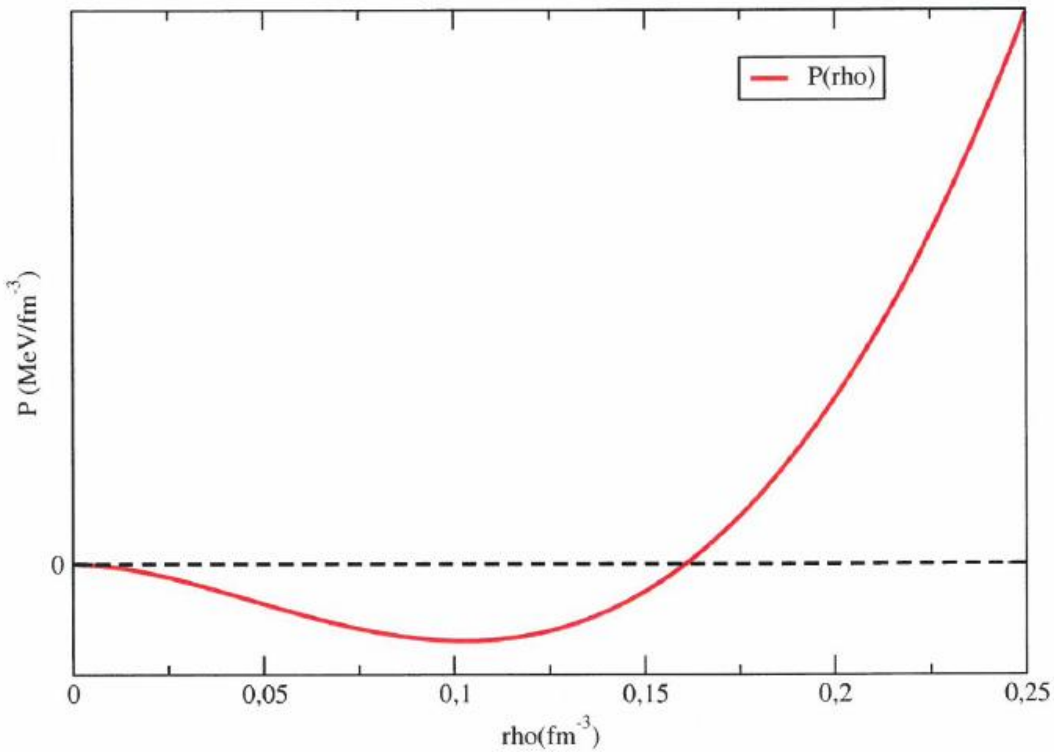
$$= \frac{1}{2} \rho \left(t_0 + \frac{1}{6} t_3 \rho^\gamma\right) \frac{3}{4}$$

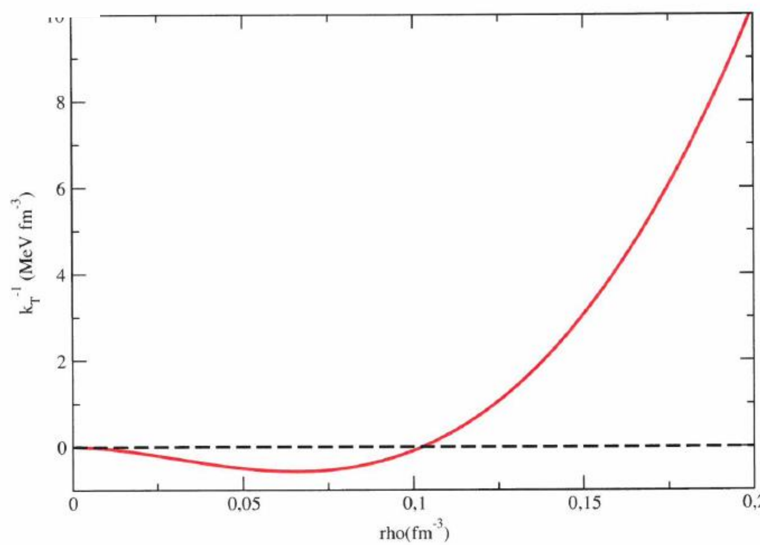


E
chemical potential
MeV
rho(fm^{-3})



P(rho)
P (MeV/fm^{-3})
rho(fm^{-3})


P(rho)
P (MeV/fm^{-3})
0
0
0,05
0,1
0,15
0,2
0,25
rho(fm^{-3})


8
6
4
2
0
k_T^{-1} (MeV fm^{-3})
0
0,05
0,1
0,15
0,2
rho(fm^{-3})