Reminder on the Wigner-Eckart theorem

In our angular momentum lectures on the Wigner-Eckart theorem we developed two equations. One for the general expectation value that depends also on the magnetic quantum numbers

$$\langle \Phi_J'| T^\mu_{\lambda_m} T^\nu_{\lambda_m'} | \Phi_J \rangle = (-1)^{J' - M} \begin{pmatrix} J & \lambda & J' \\ -M & \mu & M' \end{pmatrix} \langle \Phi_J'| T^\nu | \Phi_J \rangle,$$

and one for the reduced matrix elements in terms of

$$\langle \Phi_{J'} | T | \Phi_J \rangle = \sum_{M, \mu, M'} (-1)^{J' - M} \begin{pmatrix} J & \lambda & J' \\ -M & \mu & M' \end{pmatrix} \langle \Phi_{J'} | T^\nu_{\lambda_m} T^\nu_{\lambda_m'} | \Phi_J \rangle.$$

Why the Wigner-Eckart theorem?

Unless we have observables which depend on the magnetic quantum numbers, the degeneracy given by these quantum numbers is not seen experimentally. The typical situation when we perform shell-model calculations is that the results depend on the magnetic quantum numbers. The reason for this is that it is easy to implement the Pauli principle for many particles when we work in what we dubbed for $m$-scheme.

A resulting state in a shell-model calculations will thus depend on the total value of $M$ defined as

$$M = \sum_{j=1}^{N} m_j.$$

A shell model many-body state is given by a linear combination of Slater determinants $| \Phi_i \rangle$. That is, for some conserved quantum numbers $\lambda$ we have

$$| \Psi_c \rangle = \sum_j C_j | \Phi_j \rangle,$$

Representing a shell-model state

A general shell-model many-body state

$$| \Psi_c \rangle = \sum_j C_j | \Phi_j \rangle,$$

can be expanded as

$$| \Psi_c \rangle = C_0 | \Phi_0 \rangle + \sum_{p \neq 0} C_p | \Phi_p \rangle + \sum_{p \neq 0} \sum_{q \neq 0} C_{pq} | \Phi_{pq} \rangle + \ldots$$

Representing a shell-model state and one-body operators

A one-body operator represented by a spherical tensor of rank $\lambda$ is given as

$$O^{\lambda}_{\nu}\Phi = \sum_{p\nu} \langle \Phi | \nu_{\lambda_m} | \Phi \rangle O^{\lambda}_{\nu} \Phi,$$

meaning that when we compute a transition amplitude

$$\langle \Psi_d | O^{\lambda}_{\nu} | \Psi_c \rangle = \sum_{d \nu} C_d^{*} C_{d'} | \Phi_d | O^{\lambda}_{\nu} | \Phi_{d'} \rangle,$$

we need to compute

$$\langle \Phi_d | O^{\lambda}_{\nu} | \Phi_{d'} \rangle.$$
We want to rewrite
$$\langle \Phi_i | O_\mu^\lambda | \Phi_f \rangle,$$
in terms of the reduced matrix element only. Let us introduce the
relevant quantum numbers for the states $\Phi_i$ and $\Phi_f$. We include
only the relevant ones. We have then in $m$-scheme
$$\langle \Phi_i | O_\mu^\lambda | \Phi_f \rangle = \sum_{pq} \langle p | O_\mu^\lambda | q \rangle \langle \Phi_i | \Phi_q \rangle a_p a_q^\dagger.$$With a shell-model $m$-scheme basis it is straightforward to compute
these amplitudes. However, as mentioned above, if we wish to
related these elements to experiment, we need to use the
Wigner-Eckart theorem and express the amplitudes in terms of
reduced matrix elements.

Rewriting the transition amplitude, second step
We have
$$O_\mu^\lambda = \sum_{pq} \langle p | O_\mu^\lambda | q \rangle (1)^{j_p - j_q} a_p a_q^\dagger,$$we then single out the sum over $m_p$ and $m_q$ only and define the
recoupled one-body part of the operator as
$$\lambda^{-1} \begin{bmatrix} a_p^\dagger \alpha \beta \nu \mu \\ \end{bmatrix} = \sum_{m_p m_q} (1)^{j_p - j_q} \begin{bmatrix} j_p & \lambda & j_q \\ m_p & \mu & m_q \\ \end{bmatrix} a_p a_q^\dagger,$$with $\lambda = \sqrt{\Delta K + 1}$. This gives the following expression for the
one-body operator
$$O_\mu^\lambda = \sum_{\lambda \nu \mu} \langle p | O_\lambda^\nu \rangle \lambda^{-1} \begin{bmatrix} a_p^\dagger \alpha \beta \nu \mu \\ \end{bmatrix}.$$

Rewriting the transition amplitude, third step
With
$$O_\mu^\lambda = \sum_{pq} \langle p | O_\lambda^\nu | q \rangle \lambda^{-1} \begin{bmatrix} a_p^\dagger \alpha \beta \nu \mu \\ \end{bmatrix},$$we can write
$$\langle \Phi_i | O_\mu^\lambda | \Phi_f \rangle = \sum_{pq} \langle \Phi_i | O_\mu^\lambda | \Phi_q \rangle \langle \Phi_q | a_p^\dagger a_q | \Phi_f \rangle,$$as
$$\langle \Phi_i | O_\mu^\lambda | \Phi_f \rangle = \sum_{\lambda \nu \mu} \langle p | O_\lambda^\nu | q \rangle \lambda^{-1} \begin{bmatrix} a_p^\dagger \alpha \beta \nu \mu \\ \end{bmatrix} \langle \Phi_q | a_p^\dagger a_q | \Phi_f \rangle.$$We have suppressed the summation over quantum numbers like
$n_p, n_q$ etc.

Using the Wigner-Eckart theorem,
$$\langle \Phi_i | O_\mu^\lambda | \Phi_f \rangle = (-1)^{j_f - j_i} \begin{bmatrix} j_f & \lambda & j_i \\ M_f & \mu & M_i \\ \end{bmatrix} \langle \Phi_i | O_\lambda^\nu | \Phi_f \rangle,$$we can then define
$$\langle \Phi_i | O_\lambda^\nu | \Phi_f \rangle = \lambda^{-1} \sum_{\mu \nu \mu} \langle p | O_\nu^\lambda | q \rangle \langle \Phi_i | a_p^\dagger a_q | \Phi_f \rangle.$$The quantity to the left in the last equation is normally called the
transition amplitude or in case of a decay process, simply the decay
amplitude. The quantity $\langle \Phi_i | O_\nu^\lambda | \Phi_f \rangle$ is called the
one-body transition density while the corresponding reduced one is
simply called the reduced one-body transition density. The
transition densities characterize the many-nucleon properties of the
initial and final states. They do not carry information about the
transition operator beyond its one-body character. Finally note
that in shell-model language it would be called $\langle \Phi_i | O_\nu | \Phi_f \rangle$. 

Electromagnetic multipole moments and transitions
The reduced transition probability $B$ is defined in terms of reduced
matrix elements of a one-body operator by
$$B(i \rightarrow f) = \frac{\langle j_f | O_\lambda | j_i \rangle^2}{(2J_f + 1)}.$$With our definition of the reduced matrix element,
$$\langle j_f | O_\lambda | j_i \rangle^2 = \langle j_f | O_\lambda \rangle J_f J_i,$$the transition probability $B$ depends upon the direction of the
transition by the factor of $(2J_f + 1)$. For electromagnetic transitions
$J_i$ is that for the higher-energy initial state. But in Coulomb
excitation the initial state is usually taken as the ground state, and
it is normal to use the notation $B(1)$ for transitions from the
ground state.
Gamma transitions with $\lambda = 0$ are forbidden because the photon must carry off at least one unit of angular momentum. The $e_q$ are the electric charges for the proton and neutron in units of $e$. For the free-nucleon charge we would take $e_p - 1$ and $e_n = 0$, for the proton and neutron, respectively. Although the bare operator acts upon the protons, we will keep the general expression in terms of $e_q$ in order to incorporate the effective charges for the proton and neutron, which represent the center-of-mass corrections and the average effects of the renormalization from wavefunction admixtures outside the model space.

The most common types of transitions are $E1$, $E2$ and $M1$. The $E1$ transition operator is given by $\lambda = 1$

$$O(E1) = r_1 Y_0^0(\hat{r}) e_q e.$$ The $E2$ transition operator with $\lambda = 2$

$$O(E2) = r^2 Y_0^2(\hat{r}) e_q e.$$ The $M1$ transition operator with $\lambda = 1$ and with $\gamma^0 = 1/\sqrt{4\pi}$, we have

$$O(M1) = \sqrt{\frac{\gamma^0}{4\pi}} Y_0^1(\hat{r}) \mu e_N.$$

Gamma transitions with $\lambda = 0$ are forbidden because the photon must carry off at least one unit of angular momentum. The $e_q$ are the electric charges for the proton and neutron in units of $e$. For the free-nucleon charge we would take $e_p - 1$ and $e_n = 0$, for the proton and neutron, respectively. Although the bare operator acts upon the protons, we will keep the general expression in terms of $e_q$ in order to incorporate the effective charges for the proton and neutron, which represent the center-of-mass corrections and the average effects of the renormalization from wavefunction admixtures outside the model space.
The electromagnetic multipole moments and transitions depend upon the reduced nuclear matrix elements \( \langle f | \mathcal{O}(\lambda) | i \rangle \). These can be expressed as a sum over one-body transition densities (OBTD) times single-particle matrix elements

\[
\langle f | \mathcal{O}(\lambda) | i \rangle = \sum_{k \lambda'} \text{OBTD}(k_f k_{\lambda'} \lambda')(k_i | \mathcal{O}(\lambda) | k_{\lambda'}),
\]

where the OBTD is given by

\[
\text{OBTD}(k_f k_{\lambda'} \lambda') = \frac{\langle f | \mu_{k_f}^{\lambda'} | i \rangle}{\sqrt{(2\lambda' + 1)}}.
\]

The labels \( i \) and \( f \) are a short-hand notation for the initial and final state quantum numbers \( (n_i, j_i, \lambda_i) \) and \( (n_f, j_f, \lambda_f) \), respectively. Thus the problem is divided into two parts, one involving the nuclear structure dependent one-body transition densities OBTD, and the other involving the reduced single-particle matrix elements (SPME).
Electromagnetic multipole moments and transitions

The SPME for the orbital part of the magnetic operator is:

\[ \langle k_a | O(M_1, l) | k_b \rangle = \]

\[ - \frac{\sqrt{4\pi}}{\lambda + 1} \left( \langle k_a | Y^{(\lambda-1)}(\hat{r}) \otimes l | k_b \rangle \langle k_b | \delta_{l \lambda} | k_a \rangle \right) \mu_N \]

and the SPME simplify to:

\[ \langle k_a | O(M_1, s) | k_b \rangle = \]

\[ - \frac{\sqrt{4\pi}}{\lambda + 1} \left( \langle k_a | Y^{(\lambda-1)}(\hat{r}) \otimes s | k_b \rangle \langle k_b | \delta_{s \lambda} | k_a \rangle \right) \mu_N \]

where we have defined

\[ \langle k_a | Y^{(\lambda-1)}(\hat{r}) | k_b \rangle = \]

\[ \left( -1 \right)^{\lambda + \lambda_b + \lambda_a} \sqrt{\left( \lambda_a + 1 \right)\left( \lambda_b + 1 \right)} \]

\[ \times \left\{ j_a j_b j_a j_b \right\} \left( -1 \right)^{\lambda + \lambda_b + \lambda_a} \langle k_a | Y^{(\lambda-1)}(\hat{r}) | k_b \rangle \delta_{l \lambda} \delta_{s \lambda} \mu_N \].

For the \( M_1 \) operator the radial matrix element is

\[ < k_a | R | k_b > = \delta_{n_a n_b} \delta_{l_a l_b} \]

and the SPME simplify to:

\[ \langle k_a | O(M_1, l) | k_b \rangle = \]

\[ \left( -1 \right)^{\lambda + \lambda_b + \lambda_a} \sqrt{\left( \lambda_a + 1 \right)\left( \lambda_b + 1 \right)} \]

\[ \times \left\{ j_a j_b j_a j_b \right\} \left( -1 \right)^{\lambda + \lambda_b + \lambda_a} \langle k_a | Y^{(\lambda-1)}(\hat{r}) \otimes l | k_b \rangle \delta_{l \lambda} \delta_{s \lambda} \mu_N \].

The allowed beta decay rate \( W \) between a specific set of initial and final states is given by

\[ W_{i,f} = \left( f / K_o \right) \left[ g_N^2 R_i / F_i \right] + g_A^2 R_i / \left( GT_{s} \right) \]

where we have defined

\[ \langle k_i | Y^{(\lambda-1)}(\hat{r}) | k_f \rangle = \left( -1 \right)^{\lambda + \lambda_b + \lambda_a} \sqrt{\left( \lambda_a + 1 \right)\left( \lambda_b + 1 \right)} \]

\[ \times \left\{ j_a j_b j_a j_b \right\} \left( -1 \right)^{\lambda + \lambda_b + \lambda_a} \langle k_a | Y^{(\lambda-1)}(\hat{r}) | k_b \rangle \delta_{l \lambda} \delta_{s \lambda} \mu_N \].

where we have

\[ < s||s||s > = \sqrt{7/2} \]

and

\[ < k_a | O(M_1, l) | k_b > = \] \[ \frac{\sqrt{7}}{4\pi} \]

\[ \times \left\{ j_a j_b j_a j_b \right\} \left( -1 \right)^{\lambda + \lambda_b + \lambda_a} \sqrt{\left( \lambda_a + 1 \right)\left( \lambda_b + 1 \right)} \]

\[ \times \left\{ j_a j_b j_a j_b \right\} \left( -1 \right)^{\lambda + \lambda_b + \lambda_a} \langle k_a | Y^{(\lambda-1)}(\hat{r}) \otimes l | k_b \rangle \delta_{l \lambda} \delta_{s \lambda} \mu_N \].

The allowed beta decay rate \( W \) between a specific set of initial and final states is given by

\[ W_{i,f} = \left( f / K_o \right) \left[ g_N^2 R_i / F_i \right] + g_A^2 R_i / \left( GT_{s} \right) \]

where \( f \) is dimensionless three-body phase-space factor which depends upon the beta-decay \( Q \) value, and \( K_o \) is a specific combination of fundamental constants

\[ K_o = \frac{2\pi \hbar^2}{3 \alpha c^2} = 1.8844 \times 10^{-5} \text{erg}^2 \text{cm}^3/\text{s}. \]

The sign refers to \( \beta^- \) decay of nucleus \( (A, Z) \) into nucleus \( (A', Z') \). The weak-interaction vector \( (V) \) and axial-vector \( (A) \) coupling constants for the decay of neutron into a proton are denoted by \( g_V \) and \( g_A \), respectively.
The total decay rate for a given initial state is obtained by summing the partial rates over all final states

$$W = \sum_{f} W_{f},$$

with the branching fraction to a specific final state given by

$$b_{f} = \frac{W_{f}}{W}.$$ 

Beta decay lifetimes are usually given in terms of the half-life with a total half-life of

$$T_{1/2} = \frac{ln(2)}{W}. $$

The partial half-life for a particular final state will be denoted by

$$t_{1/2} = \frac{T_{1/2}}{W}.$$ 

Historically one combines the partial half-life for a particular decay with the calculated phase-space factor $f$ to obtain an $ft$ value given by

$$ft_{1/2} = \frac{C}{B(F^{\pm}) + (g_{A}/g_{V})^{2}B(GT^{\pm})},$$

where

$$C = \frac{ln(2)K_{o}}{(g_{V})^{2}}.$$ 

One often compiles the allowed beta decay in terms of a log $ft$ which stands for log of the $ft_{1/2}$ value. The values of the coupling constants for Fermi decay, $g_{V}$, and Gamow-Teller decay, $g_{A}$, are obtained as follows. For a $0^{+} \rightarrow 0^{+}$ nuclear transition $B(GT) = 0$, and for a transition between $T = 1$ analogue states with $B(F^{\pm}) = 2$ we find

$$C = 2t_{1/2}f.$$ 

The partial half-lives and $Q$ values for several $0^{+} \rightarrow 0^{+}$ analogue transitions have been measured to an accuracy of about one part in 10000. With phase space factors one obtains

$$C = 6170(4).$$ 

This result, together with the value of $K_{o}$ can be used to obtain $g_{V}$.

At the quark level $g_{V} = -g_{A}$. But for nuclear structure we use the value obtained from the neutron to proton beta decay

$$|g_{A}/g_{V}| = 1.261(8).$$

The matrix elements obey the triangle conditions $J_{f} = J_{i}$ ($\Delta J = 0$). The Fermi operator has $\pi_{F} = +1$, and thus the initial and final nuclear states must have $\pi_{F} = +1$ for the matrix element to be nonzero under the parity transform. When isospin is conserved the Fermi matrix element must obey the isospin triangle condition $T_{f} = T_{i}$ ($\Delta T = 0$), and the Fermi operator can only connect isobaric analogue states.
For β⁻ decay
\[ T_{-|ω, J, M, T, T_ω⟩} = \sqrt{(T(T_ω + 1) - T_ω(T_ω - 1)ω, J, M, T, T_ω - 1)}, \]
and
\[ B(F⁻) = |⟨ωf, Jf, Mf, Tf, T_ω + 1|T⁻|ωi, Ji, Mi, Ti, T_ω⟩|^2 = \frac{1}{3} \sum (T(T_ω + 1) - T_ω(T_ω - 1)). \]

For β⁺ we have
\[ B(F⁺) = |⟨ωf, Jf, Mf, Tf, T_ω + 1|T⁺|ωi, Ji, Mi, Ti, T_ω⟩|^2 = \frac{1}{3} \sum (T(T_ω + 1) - T_ω(T_ω - 1)). \]

The sum rule for Gamow-Teller is obtained as follows
\[ \sum |⟨f |T⁺|i⟩|^2 - \sum |⟨f |T⁻|i⟩|^2 = 3⟨i|2T_ω|i⟩ = 3(N_i - Z_i). \]

The matrix elements of s has the selection rules \( δ_{ℓa, ℓb} \) and \( δ_{na, nb} \). Thus the orbits which are connected by the GT operator are very selective; they are those in the same major oscillator shell with the same \( f \) value. The matrix elements such as \( 5d_3/2 - 0d_5/2 \) which have the allowed \( ΔJ \) coupling but are zero due to the \( Δℓ \) coupling are called \( f \)-forbidden matrix elements.

The sum rule for Fermi is obtained from the sum
\[ \sum |⟨f |T⁻|i⟩|^2 - \sum |⟨f |T⁺|i⟩|^2 = 3⟨i|T_ω|i⟩ = 3(N_i - Z_i). \]
We have used the fact that \( \sigma_z^2 = \sigma_x^2 = \sigma_y^2 = 1 \). When \( k \neq k' \) the operators commutes and cancel. Thus
\[
\sum_r [\hat{N}_0(F_{-i}) - \hat{N}_0(F_{i})] = (N - Z).
\]
and
\[
\sum_r [\hat{N}_0(G_{T+}) - \hat{N}_0(G_{T-})] = 3(N - Z).
\]
The sum-rule for the Fermi matrix elements applies even when isospin is not conserved.

For \( N > Z \) we usually have \( T_i - T_{zi} \) which means that \( B(F_{-i}) = 0 \). For \( N = Z \) \( T_i = 0 \) and \( T_{zi} = 0 \) we get \( B(F_{-i}) = 0 \), and for \( N > Z \) we have \( B(F_{-i}) = B(F_{i}) = 2 \). Fermi transitions which would be zero if isospin is conserved are called isospin-forbidden Fermi transitions.

When \( N > Z \) there are some situations where one has \( B(G_{T+}) = 0 \), and then we obtain \( B(G_{T-}) = 3(N - Z) \). In particular for the \( \beta \)-decay of the neutron we have \( B(F_{-i}) = 1 \) and \( B(G_{T-}) = 3 \).

The state \( \langle \omega \rangle \) is an exact result. We call this quantity the correlation energy.

The Schrödinger equation is
\[
\hat{H}|\Psi_0\rangle = E|\Psi_0\rangle,
\]
and multiplying the latter from the left with \( \langle \Phi_0 | \) gives
\[
\langle \Phi_0 | \hat{H} |\Psi_0\rangle = E \langle \Phi_0 | \Psi_0 \rangle - E,
\]
and subtracting from this equation
\[
\langle \Phi_0 | \hat{H} - W_0 | \Phi_0 \rangle = W_0 \langle \Phi_0 | \Phi_0 \rangle - W_0,
\]
and using the fact that the both operators \( \hat{H} \) and \( \hat{H}_0 \) are hermitian results in
\[
\Delta E = E - W_0 = - \langle \Phi_0 | \hat{H}_0 | \Phi_0 \rangle,
\]
which is an exact result. We call this quantity the correlation energy.

This equation forms the starting point for all perturbative derivations. However, as it stands it represents nothing but a mere formal rewriting of Schrodinger’s equation and is not of much practical use. The exact wave function \( \langle \psi_0 | \) is unknown. In order to obtain a perturbative expansion, we need to expand the exact wave function in terms of the interaction \( \hat{H}_0 \).

Here we have assumed that our model space defined by the operator \( \hat{P} \) is one-dimensional, meaning that
\[
\hat{P} = |\Phi_0 \rangle \langle \Phi_0 |
\]
and
\[
\hat{Q} = \sum_{m=1} |\Phi_m \rangle \langle \Phi_m |
\]

We can thus rewrite the exact wave function as
\[
|\psi_0 \rangle = (\hat{P} + \hat{Q}) |\psi_0 \rangle = |\Phi_0 \rangle + \hat{Q} |\psi_0 \rangle.
\]

Going back to the Schrödinger equation, we can rewrite it as, adding and subtracting a term \( \omega |\psi_0 \rangle \) as
\[
\left( \omega - \hat{H} \right) |\psi_0 \rangle = \left( \omega - E + \hat{H}_0 \right) |\psi_0 \rangle.
\]
where \( \omega \) is an energy variable to be specified later.
We assume also that the resolvent of \((\omega - \hat{H}_0)\) exits, that is it has an inverse which defined the unperturbed Green’s function as

\[
(\omega - \hat{H}_0)^{-1} = \frac{1}{\omega - \hat{H}_0}
\]

We can rewrite Schroedinger’s equation as

\[
|\Psi_0\rangle = \frac{1}{\omega - \hat{H}_0} \left(\omega - E + \hat{H}_0\right) |\Psi_0\rangle,
\]

and multiplying from the left with \(\hat{Q}\) results in

\[
\hat{Q}|\Psi_0\rangle = \frac{\hat{Q}}{\omega - \hat{H}_0} \left(\omega - E + \hat{H}_0\right) |\Psi_0\rangle,
\]

which is possible since we have defined the operator \(\hat{Q}\) in terms of the eigenfunctions of \(\hat{H}\).

These operators commute meaning that

\[
\hat{Q} = \frac{1}{\omega - \hat{H}_0} \left(\omega - E + \hat{H}_0\right)
\]

With these definitions we can in turn define the wave function as

\[
|\Psi_0\rangle = |\Phi_0\rangle + \frac{\hat{Q}}{\omega - \hat{H}_0} \left(\omega - E + \hat{H}_0\right) |\Psi_0\rangle.
\]

This equation is again nothing but a formal rewrite of Schrödinger’s equation and does not represent a practical calculational scheme. It is a non-linear equation in two unknown quantities, the energy \(E\) and the exact wave function \(|\Psi_0\rangle\). We can however start with a guess for \(|\Psi_0\rangle\) on the right hand side of the last equation.

The most common choice is to start with the function which is expected to exhibit the largest overlap with the wave function we are searching after, namely \(|\Phi_0\rangle\). This can again be inserted in the solution for \(|\Psi_0\rangle\) in an iterative fashion and if we continue along these lines we end up with

\[
|\Psi_0\rangle = \sum_{n=0}^{\infty} \left\{ \frac{\hat{Q}}{\omega - \hat{H}_0} \left(\omega - E + \hat{H}_0\right) \right\}^n |\Phi_0\rangle,
\]

for the wave function and

\[
\Delta E = \sum_{n=0}^{\infty} \Phi_0 |\Phi_0\rangle \left\{ \frac{\hat{Q}}{\omega - \hat{H}_0} \left(\omega - E + \hat{H}_0\right) \right\}^n |\Phi_0\rangle,
\]

which is now a perturbative expansion of the exact energy in terms of the interaction \(\hat{H}\) and the unperturbed wave function \(|\Psi_0\rangle\).

In our equations for \(|\Psi_0\rangle\) and \(\Delta E\) in terms of the unperturbed solutions \(|\Phi_0\rangle\) we have still an undetermined parameter \(\omega\) and a dependency on the exact energy \(E\). Not much has been gained thus from a practical calculational point of view.

In Brillouin-Wigner perturbation theory it is customary to set \(\omega = E\). This results in the following perturbative expansion for the energy \(\Delta E\)

\[
\Delta E = \sum_{n=0}^{\infty} \Phi_0 |\Phi_0\rangle \left\{ \frac{\hat{Q}}{E - \hat{H}_0} \left(E - \hat{H}_0\right) \right\}^n |\Phi_0\rangle =
\]

\[
|\Phi_0\rangle \left(\hat{H}_0 + \hat{H} \frac{\hat{Q}}{E - \hat{H}_0} \hat{H} + \hat{H} \frac{\hat{Q}}{E - \hat{H}_0} \hat{H} + \ldots \right) |\Phi_0\rangle.
\]

These operators commute meaning that

\[
\hat{Q} = \frac{1}{E - \hat{H}_0} \left(E - \hat{H}_0\right)
\]

With these definitions we can in turn define the wave function as

\[
|\Psi_0\rangle = |\Phi_0\rangle + \frac{\hat{Q}}{E - \hat{H}_0} \left(E - \hat{H}_0\right) |\Psi_0\rangle.
\]

This equation is again nothing but a formal rewrite of Schrödinger’s equation and does not represent a practical calculational scheme. It is a non-linear equation in two unknown quantities, the energy \(E\) and the exact wave function \(|\Psi_0\rangle\). We can however start with a guess for \(|\Psi_0\rangle\) on the right hand side of the last equation.

Defining \(e = E - \hat{H}_0\) and recalling that \(\hat{H}_0\) commutes with \(\hat{Q}\) by construction and that \(\hat{Q}\) is an idempotent operator \(\hat{Q}^2 = \hat{Q}\). Using this equation in the above expansion for \(\Delta E\) we can write the denominator

\[
\hat{Q}^{-1} = \frac{1}{e - \hat{H}_0}\n\]

Inserted in the expression for \(\Delta E\) leads to

\[
\Delta E = \langle \Phi_0 | \hat{H}_0 + \hat{H} \frac{\hat{Q}}{E - \hat{H}_0} \hat{H} + \hat{H} \frac{\hat{Q}}{E - \hat{H}_0} \hat{H} + \ldots | \Phi_0 \rangle.
\]

In RS perturbation theory we set \(\omega = W_0\) and obtain the following expression for the energy difference

\[
\Delta E = \sum_{n=0}^{\infty} \langle \Phi_0 \rangle \left(\hat{H}_0 + \hat{H} \frac{\hat{Q}}{W_0 - \hat{H}_0} \hat{H} + \hat{H} \frac{\hat{Q}}{W_0 - \hat{H}_0} \hat{H} + \ldots \right) |\Phi_0\rangle -
\]

\[
|\Phi_0\rangle \left(\hat{H}_0 + \hat{H} \frac{\hat{Q}}{W_0 - \hat{H}_0} \hat{H}_0 \hat{\Delta E} + \hat{H} \frac{\hat{Q}}{W_0 - \hat{H}_0} \hat{H}_0 \hat{\Delta E} + \hat{H} \frac{\hat{Q}}{W_0 - \hat{H}_0} \hat{H}_0 \hat{\Delta E} + \ldots \right) |\Phi_0\rangle.
\]
Recalling that $\hat{Q}$ commutes with $\hat{H}_0$ and since $\Delta E$ is a constant we obtain that

$$\hat{Q}\Delta E|\Phi_0\rangle - \hat{Q}\Delta E|\Phi_0\rangle = 0.$$ 

Inserting this results in the expression for the energy results in

$$\Delta E = \langle \Phi_0 | (\hat{H}_I + \hat{H}_I \hat{Q} W_0 - \hat{H}_0 \hat{H}_I + \hat{H}_I \hat{Q} W_0 - \hat{H}_0 (\hat{H}_I - \Delta E) \hat{Q} W_0 - \hat{H}_0 \hat{H}_I + \ldots) |\Phi_0\rangle.$$ 

We can now this expression in terms of a perturbative expression in terms of $\hat{H}_I$ where we iterate the last expression in terms of $\Delta E$

$$\Delta E = \sum_{i=1}^{\infty} \Delta E^{(i)}.$$ 

We get the following expression for $\Delta E^{(i)}$

$$\Delta E^{(i)} = \langle \Phi_0 | \hat{H}_I | \Phi_0\rangle,$$

which is just the contribution to first order in perturbation theory,

$$\Delta E^{(2)} = \langle \Phi_0 | \hat{H}_I \hat{Q} W_0 - \hat{H}_0 \hat{H}_I \hat{Q} W_0 - \hat{H}_0 \hat{Q}|\Phi_0\rangle,$$

which is the contribution to second order.

$$\Delta E^{(3)} = \langle \Phi_0 | \hat{H}_I \hat{Q} W_0 - \hat{H}_0 \hat{H}_I \hat{Q} W_0 - \hat{H}_0 \hat{Q} \hat{H}_I |\Phi_0\rangle - \langle \Phi_0 | \hat{H}_I \hat{Q} W_0 - \hat{H}_0 \hat{H}_I \hat{Q} W_0 - \hat{H}_0 \hat{Q}|\Phi_0\rangle,$$

being the third-order contribution.

I hope this is not the case

Learning outcomes

Topics we have covered this year

- Single-particle properties and mean-field and relation to data
- How to set up basis states in second quantization and find expectation values
- Angular momentum properties and the Wigner-Eckart theorem
- Short survey of properties of nuclear forces
- The nuclear shell model
- And how to relate a shell-model calculation to decays and properties of decays.

Final presentation

- Introduction with motivation
- Explain an eventual experimental set up
- Give a short overview of the theory employed and how it relates to the analysis of eventual data
- Present and discuss your results
- Summary, conclusions and perspectives
- Anything else you think is important. Useful to have backup slides

In total your talk should have a duration of 20-25 minutes, but longer is also ok.
What? Me worry?

Best wishes to you all and thanks so much for your efforts this semester.